Math 211

Lecture #7

September 19, 2000

Modelling Equations

- Two ways to write the rate of change of something, e.g., of a population P
  - ◇ The mathematical way is the derivative:
  $$\frac{dP}{dt}.$$
  - ◇ The scientific way involves modelling:
  $$r\left(1 - \frac{P}{K}\right)P.$$

Modelling Equations

- Setting the two equal gives a differential equation model
$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P$$
- An equation says that two distinct mathematical expressions are equal.

Estimating Parameters

- Malthusian model $P' = rP$

$$P(t) = P_0 e^{rt}$$

  - ◇ Two parameters P_0 and r.
  - ◇ Two measurements or observations needed to find the values of P_0 and r.
  - ◇ It is better to use all of the data and use least squares (linear regression).

Estimating Parameters

- Logistic model $P' = r(1 - P/K)P$

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

  - ◇ Three parameters, P_0, r, and K.
  - ◇ Three measurements or observations needed to find the values of P_0, r, and K.
  - ◇ It is better to use all of the data and use least squares. (Nonlinear regression)

Logistic Model

- Does a very good job of modeling the growth of populations under controlled circumstances.
  - ◇ In laboratory experiments.
  - ◇ In other circumstances when the situation does not change.

Logistic Model

- For human populations the model always breaks down.
  - ◇ Other factors become important, such as immigration.
  - ◇ Advance of technology.
  - ◇ Changing habits of life.

Compound Interest

- Put some money into an account that returns a percentage each year, compounded continuously. How much is there some time later?
  - ◇ "Some money" is P_0 measured in \$1000.
  - ◇ "Returns a percentage" is $r\%$/year.
  - ◇ "Some time later" is measured in years.
  - ◇ "Compounded continuously" means

$$P' = rP.$$

Compound Interest

- Solution

$$P(t) = P_0 e^{rt}$$

- The principal grows exponentially.
- If $r = 8\%$, then after 20 years

$$\begin{aligned} P(20) &= P_0 e^{0.08\times 20} \\ &= 4.953\, P_0 \end{aligned}$$

- After 40 years $P(40) = 24.5325\, P_0$.

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Retirement Account

- Set up a retirement account by investing an initial amount. In addition, deposit a fixed amount each year until you retire. Assume it returns a percentage each year, compounded continuously. How much is there some time later?
  - "A fixed amount each year" is D, measured in $1,000 each year. We assume this is invested continuously.

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Retirement Account

- The model is

$$P' = rP + D.$$

- Solution

$$P(t) = P_0 e^{rt} + \frac{D}{r}[e^{rt} - 1].$$

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Retirement Acount

- Suppose you start with an investment of $1,000 at the age of 25, and invest $100 each month until you retire at 65. The account returns 8% per year. How much is in the retirement account when you retire?
- $377,521.
- Is this enough to retire on?

Retirement Planning

- If you need a certain income after you retire, how much must you have in your retirement account when you retire?
  - ◇ "Certain income" is I (in \$1000/year) withdrawn from the account.
  - ◇ "How much" is the amount P_0 in the account at retirement.
  - ◇ The account still grows due to its return at r%/year.

Retirement Planning

- The model is

$$P' = rP - I.$$

- Solution

$$P(t) = P_0 e^{rt} - \frac{I}{r}[e^{rt} - 1].$$

Retirement Planning

- If you will need an income of \$75,000 for 30 years after retirement and your account returns 6%, your account balance at retirement should be

$$\$1,043,000.$$

Retirement Planning

- Instead of investing a fixed amount each month, it would be more realistic to invest a percentage of your salary. What should this percentage be in order to accumulate an adequate investment balance?
- You starting salary is S_0.
- Assume it will increase at $s\%$ per year.
  - ◇ Then $S' = sS$, or $S(t) = S_0 e^{st}$.

Retirement Planning

- The model for the growth of the retirement account is

$$P' = rP + \lambda S_0 e^{st} \quad \text{with} \quad P(0) = P_0.$$

- Solution

$$P(t) = P_0 e^{rt} + \frac{\lambda S_0}{r - s}\left[e^{rt} - e^{st}\right].$$

Retirement Planning

- Assume
  - ◇ $P_0 =$ \$1,000 and $r = 8\%$
  - ◇ $S_0 =$ \$35,000 and $s = 4\%$
    - ⋆ Notice that $S(40) =$ \$173,356.
  - ◇ Need a retirement income of \$150,000.
    - ⋆ Aim for a balance at retirement of \$2,000,000.
- Requires $\lambda = 11.53\%$.

Other Strategies

- Delayed gratification. Deposit a percentage of your salary that starts at λ%, and decays linearly to 0 over 40 years.

$$P' = rP + \lambda(1 - t/40)S_0 e^{st}$$

- Immediate gratification. Deposit a percentage of your salary that starts at 0 and grow linearly over 40 years to λ%.

$$P' = rP + \frac{\lambda t}{40} S_0 e^{st}$$