

# Math 211

Lecture #7

September 19, 2000

# Modelling Equations

- Two ways to write the rate of change of something, e.g., of a population  $P$ 
  - ◇ The mathematical way is the derivative:

$$\frac{dP}{dt}.$$

- ◇ The scientific way involves modelling:

$$r \left( 1 - \frac{P}{K} \right) P.$$

# Modelling Equations

- Setting the two equal gives a differential equation model

$$\frac{dP}{dt} = r \left( 1 - \frac{P}{K} \right) P$$

- An equation says that two distinct mathematical expressions are equal.

# Estimating Parameters

- Malthusian model  $P' = rP$

$$P(t) = P_0 e^{rt}$$

- ◇ Two parameters  $P_0$  and  $r$ .
- ◇ Two measurements or observations needed to find the values of  $P_0$  and  $r$ .
- ◇ It is better to use all of the data and use least squares (linear regression).

# Estimating Parameters

- Logistic model  $P' = r(1 - P/K)P$

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

- ◇ Three parameters,  $P_0$ ,  $r$ , and  $K$ .
- ◇ Three measurements or observations needed to find the values of  $P_0$ ,  $r$ , and  $K$ .
- ◇ It is better to use all of the data and use least squares. (Nonlinear regression)

# Logistic Model

- Does a very good job of modeling the growth of populations under controlled circumstances.
  - ◇ In laboratory experiments.
  - ◇ In other circumstances when the situation does not change.

# Logistic Model

- For human populations the model always breaks down.
  - ◇ Other factors become important, such as immigration.
  - ◇ Advance of technology.
  - ◇ Changing habits of life.

# Compound Interest

- Put some money into an account that returns a percentage each year, compounded continuously. How much is there some time later?
  - ◇ “Some money” is  $P_0$  measured in \$1000.
  - ◇ “Returns a percentage” is  $r\%/year$ .
  - ◇ “Some time later” is measured in years.
  - ◇ “Compounded continuously” means

$$P' = rP.$$

# Compound Interest

- Solution

$$P(t) = P_0 e^{rt}$$

- The principal grows exponentially.
- If  $r = 8\%$ , then after 20 years

$$\begin{aligned} P(20) &= P_0 e^{0.08 \times 20} \\ &= 4.953 P_0 \end{aligned}$$

- After 40 years  $P(40) = 24.5325 P_0$ .

# Retirement Account

- Set up a retirement account by investing an initial amount. In addition, deposit a fixed amount each year until you retire. Assume it returns a percentage each year, compounded continuously. How much is there some time later?
  - ◇ “A fixed amount each year” is  $D$ , measured in \$1,000 each year. We assume this is invested continuously.

# Retirement Account

- The model is

$$P' = rP + D.$$

- Solution

$$P(t) = P_0 e^{rt} + \frac{D}{r} [e^{rt} - 1].$$

## Retirement Account

- Suppose you start with an investment of \$1,000 at the age of 25, and invest \$100 each month until you retire at 65. The account returns 8% per year. How much is in the retirement account when you retire?
- \$377,521.
- Is this enough to retire on?

# Retirement Planning

- If you need a certain income after you retire, how much must you have in your retirement account when you retire?
  - ◇ “Certain income” is  $I$  (in \$1000/year) withdrawn from the account.
  - ◇ “How much” is the amount  $P_0$  in the account at retirement.
  - ◇ The account still grows due to its return at  $r\%/year$ .

# Retirement Planning

- The model is

$$P' = rP - I.$$

- Solution

$$P(t) = P_0 e^{rt} - \frac{I}{r} [e^{rt} - 1].$$

# Retirement Planning

- If you will need an income of \$75,000 for 30 years after retirement and your account returns 6%, your account balance at retirement should be

\$1,043,000.

# Retirement Planning

- Instead of investing a fixed amount each month, it would be more realistic to invest a percentage of your salary. What should this percentage be in order to accumulate an adequate investment balance?
- Your starting salary is  $S_0$ .
- Assume it will increase at  $s\%$  per year.
  - ◇ Then  $S' = sS$ , or  $S(t) = S_0e^{st}$ .

# Retirement Planning

- The model for the growth of the retirement account is

$$P' = rP + \lambda S_0 e^{st} \quad \text{with} \quad P(0) = P_0.$$

- Solution

$$P(t) = P_0 e^{rt} + \frac{\lambda S_0}{r - s} [e^{rt} - e^{st}].$$

# Retirement Planning

- Assume
  - ◇  $P_0 = \$1,000$  and  $r = 8\%$
  - ◇  $S_0 = \$35,000$  and  $s = 4\%$ 
    - ★ Notice that  $S(40) = \$173,356$ .
  - ◇ Need a retirement income of \$150,000.
    - ★ Aim for a balance at retirement of \$2,000,000.
- Requires  $\lambda = 11.53\%$ .

# Other Strategies

- Delayed gratification. Deposit a percentage of your salary that starts at  $\lambda\%$ , and decays linearly to 0 over 40 years.

$$P' = rP + \lambda(1 - t/40)S_0e^{st}$$

- Immediate gratification. Deposit a percentage of your salary that starts at 0 and grow linearly over 40 years to  $\lambda\%$ .

$$P' = rP + \frac{\lambda t}{40}S_0e^{st}$$