

Math 211

Complex Numbers and Matrices

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Complex Numbers

A *complex number* is one of the form $z = x + iy$, where x and y are real numbers.

- Geometric representation — the complex plane.
 - ♦ $z = x + iy \leftrightarrow (x, y)$.
- x is the *real part* of z ; $x = \operatorname{Re}z$.
- y is the *imaginary part* of z ; $y = \operatorname{Im}z$.
 - ♦ The imaginary part of the complex number $z = x + iy$ is the *real number* y .
- Addition and multiplication ($i^2 = -1$).

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Complex Conjugate

Definition: The *conjugate* of $z = x + iy$ is $\bar{z} = x - iy$.

- $z = \bar{z} \Leftrightarrow z$ is a real number.
- $x = \operatorname{Re}z = \frac{z + \bar{z}}{2}$; $y = \operatorname{Im}z = \frac{z - \bar{z}}{2i}$
- $\overline{z + w} = \bar{z} + \bar{w}$; $\overline{z - w} = \bar{z} - \bar{w}$
- $\overline{zw} = \bar{z} \cdot \bar{w}$; $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

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Absolute Value

Definition: The *absolute value* of $z = x + iy$ is the real number $|z| = \sqrt{x^2 + y^2}$.

- $z \cdot \bar{z} = |z|^2 = x^2 + y^2$.
- $|zw| = |z||w|$
- $\left| \frac{z}{w} \right| = \frac{|z|}{|w|}$
- $|z + w| \leq |z| + |w|$

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Quotients

- The reciprocal of $z = x + iy$

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

- The quotient

$$\frac{z}{w} = z \cdot \frac{1}{w} = z \cdot \frac{\bar{w}}{|w|^2} = \frac{z\bar{w}}{|w|^2}$$

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 $z\bar{z}$

Polar Representation

- $z = x + iy = r[\cos \theta + i \sin \theta]$.
 - ♦ θ is called the *argument* of z .
 - ▶ $\tan \theta = y/x$.
 - ♦ $r = |z|$.
- *Euler's formula:* $e^{i\theta} = \cos \theta + i \sin \theta$.
 - ♦ $z = |z|e^{i\theta}$.
 - ♦ $\bar{z} = |z|e^{-i\theta}$.

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Multiplication

- Two complex numbers

$$z = |z|e^{i\theta} \quad \text{and} \quad w = |w|e^{i\phi}$$

- The product is

$$zw = |z|e^{i\theta} \cdot |w|e^{i\phi} = |z||w|e^{i(\theta+\phi)}.$$

- ♦ The absolute value of the product zw is the product of the absolute values of z and w : $|zw| = |z||w|$.
- ♦ The argument of the product zw is the sum of the arguments of z and w .

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Complex Exponential

Definition: For $z = x + iy$ we define

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x [\cos y + i \sin y].$$

Properties:

- $e^{z+w} = e^z \cdot e^w$; $e^{z-w} = e^z \cdot e^{-w} = e^z / e^w$
- $\overline{e^z} = e^{\bar{z}}$
- $|e^z| = e^x = e^{\operatorname{Re}z}$
- If λ is a complex number, then $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$

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Complex Matrices

Matrices (or vectors) with complex entries inherit many of the properties of complex numbers.

- $M = A + iB$ where $A = \operatorname{Re}M$ and $B = \operatorname{Im}M$ are real matrices.
- $\overline{\overline{M}} = M$; $M = \overline{M} \Leftrightarrow M$ is real.
- $\operatorname{Re}M = \frac{1}{2}(M + \overline{M})$; $\operatorname{Im}M = \frac{1}{2i}(M - \overline{M})$
- $\overline{M + N} = \overline{M} + \overline{N}$
- $\overline{Mz} = \overline{M}\overline{z}$

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