Math 211

Complex Numbers and Matrices

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Complex Numbers

A *complex number* is one of the form $z = x + iy$, where x and y are real numbers.

- Geometric representation — the complex plane.
  - $z = x + iy \;\leftrightarrow\; (x, y)$.
- x is the *real part* of z; $x = \mathrm{Re} z$.
- y is the *imaginary part* of z; $y = \mathrm{Im} z$.
  - The imaginary part of the complex number $z = x + iy$ is the *real number* y.
- Addition and multiplication $(i^2 = -1)$.

Complex Conjugate

Definition: The *conjugate* of $z = x + iy$ is $\overline{z} = x - iy$.

- $z = \overline{z} \Leftrightarrow z$ is a real number.
- $x = \mathrm{Re} z = \dfrac{z + \overline{z}}{2}$; $y = \mathrm{Im} z = \dfrac{z - \overline{z}}{2i}$
- $\overline{z + w} = \overline{z} + \overline{w}$; $\quad \overline{z - w} = \overline{z} - \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$; $\quad \overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

Return

Absolute Value

Definition: The *absolute value* of $z = x + iy$ is the real number $|z| = \sqrt{x^2 + y^2}$.

- $z \cdot \overline{z} = |z|^2 = x^2 + y^2$.
- $|zw| = |z||w|$
- $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$
- $|z + w| \leq |z| + |w|$

Quotients

- The reciprocal of $z = x + iy$

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}.$$

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

- The quotient

$$\frac{z}{w} = z \cdot \frac{1}{w} = z \cdot \frac{\overline{w}}{|w|^2} = \frac{z\overline{w}}{|w|^2}$$

Polar Representation

- $z = x + iy = r[\cos\theta + i\sin\theta]$.
  - θ is called the *argument* of z.
    - $\tan\theta = y/x$.
  - $r = |z|$.
- *Euler's formula:* $e^{i\theta} = \cos\theta + i\sin\theta$.
  - $z = |z|e^{i\theta}$.
  - $\overline{z} = |z|e^{-i\theta}$.

Multiplication

- Two complex numbers

$$z = |z|e^{i\theta} \quad \text{and} \quad w = |w|e^{i\phi}$$

- The product is

$$zw = |z|e^{i\theta} \cdot |w|e^{i\phi} = |z||w|e^{i(\theta+\phi)}.$$

  - The absolute value of the product zw is the product of the absolute values of z and w: $|zw| = |z||w|$.
  - The argument of the product zw is the sum of the arguments of z and w.

Return

Complex Exponential

Definition: For $z = x + iy$ we define

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x[\cos y + i \sin y].$$

Properties:

- $e^{z+w} = e^z \cdot e^w; \quad e^{z-w} = e^z \cdot e^{-w} = e^z/e^w$
- $\overline{e^z} = e^{\overline{z}}$
- $|e^z| = e^x = e^{\mathrm{Re}\, z}$
- If λ is a complex number, then $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$

Complex Matrices

Matrices (or vectors) with complex entries inherit many of the properties of complex numbers.

- $M = A + iB$ where $A = \mathrm{Re}M$ and $B = \mathrm{Im}M$ are real matrices.
- $\overline{M} = A - iB; \quad M = \overline{M} \Leftrightarrow M$ is real.
- $\mathrm{Re}M = \frac{1}{2}(M + \overline{M}); \quad \mathrm{Im}M = \frac{1}{2i}(M - \overline{M})$
- $\overline{M + N} = \overline{M} + \overline{N}$
- $\overline{M\mathbf{z}} = \overline{M}\overline{\mathbf{z}}$