Math 211 Sec. 4

Linear Equations

Mixing Problems

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Linear Equations

A *linear equation* is one of the form

$$x' = a(t)x + f(t).$$

- Example: $x' = \tan(t)x + 3\sin^2(t)$
- The unknown function x and its derivative must appear *linearly*.
- The equation is *homogeneous* if $f = 0$
  - $x' = a(t)x, \quad$ e.g. $\quad x' = \tan(t)x$
- The equation is *inhomogeneous* if $f \neq 0$

Homogenous Linear Equations

- Homogeneous linear equations are separable.

$$\begin{aligned} \frac{dx}{dt} &= a(t)x \\ \frac{dx}{x} &= a(t)\,dt \\ \ln|x(t)| &= \int a(t)\,dt \\ x(t) &= Ae^{\int a(t)\,dt} \end{aligned}$$

Example: $x' = \tan(t)x$.

$$\begin{aligned} x(t) &= Ae^{\int \tan(t)\,dt} \\ &= Ae^{-\ln(\cos(t))} \\ &= \frac{A}{\cos t} \\ &= A\sec t \end{aligned}$$

Example: $x' = \tan(t)x + 3\sin^2(t)$

- Rewrite as $\quad x' - \tan(t)x = 3\sin^2(t)$
- Multiply by $\cos t$.

$$\cos(t)x' - \sin(t)x = 3\sin^2(t)\cos(t)$$

The left hand side is the derivative of $\cos(t)x$. So

$$[\cos(t)x]' = 3\sin^2(t)\cos(t)$$

- Integrate

$$\cos(t)x(t) = 3\int \sin^2(t)\cos(t)\,dt = \sin^3(t) + C$$

- Solve for x

$$x(t) = \frac{\sin^3(t) + C}{\cos(t)}.$$

How did we do that? Can we do it in general?

The Key Step for $x' = ax + f$

- Rewrite as $x' - ax = f$.
- Multiply by a function $u(t)$ so that

$$u[x' - ax] = [ux]'$$

$$ux' - aux = ux' + u'x$$

♦ True if $u' = -au$. Linear, homogeneous

$$u(t) = e^{-\int a(t)\,dt} \quad \text{is one solution.}$$

♦ u is called an *integrating factor.*

Solving the Linear Equation

$$x' = a(t)x + f(t)$$

Four step process:

1. Rewrite as $x' - ax = f$.
2. Multiply by the integrating factor

$$u(t) = e^{-\int a(t)\,dt}.$$

Equation becomes $[ux]' = ux' - aux = uf$.

3. Integrate: $u(t)x(t) = \int u(t)f(t)\,dt + C$.
4. Solve for $x(t)$.

Examples

- $x' = -4x + 8, \quad x(0) = 0.$
- $x' = 2tx + e^{t^2}, \quad x(0) = 1.$
- $y' = 3y - t, \quad y(0) = 2.$
- $z' = (z + 1)\cos t, \quad z(\pi) = -1.$

Mixing Problem #1

A tank originally holds 500 gallons of pure water. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 5 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.

Model

- $S(t) =$ the amount of sugar in the tank in lbs.
- *Concentration* $=$ pounds per unit volume.
  - $c(t) = \dfrac{S(t)}{V} \dfrac{\text{lbs}}{\text{gal}}.$
- Modeling is easier in terms of the total amount, $S(t)$.
- Draw a picture.

Balance Law

- Rate of change = Rate in - Rate out
- Rate = volume rate $\times$ concentration
- For the problem
  - Rate in $= 5\frac{\text{gal}}{\text{min}} \times \frac{1}{2}\frac{\text{lb}}{\text{gal}} = 2.5\frac{\text{lb}}{\text{min}}$
  - Rate out $= 5\frac{\text{gal}}{\text{min}} \times \frac{S}{500}\frac{\text{lb}}{\text{gal}} = \frac{S}{100}\frac{\text{lb}}{\text{min}}$

Solution

$$\frac{dS}{dt} = \text{Rate in - Rate out}$$
$$= 2.5 - \frac{S}{100}.$$

- General solution: $S(t) = 250 + Ce^{-t/100}$.
- Particular solution: $S(t) = 250(1 - e^{-t/100})$.

Other possible initial conditions

- There is initially 20 lbs of sugar in the tank.
- The concentration of sugar in the tank at $t = 0$ is 1 lb/gallon.

Mixing Problem #2

A tank originally holds 500 gallons of sugar water with a concentration of $\frac{1}{10}$ lb/gal. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 10 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.

Return

Solution

- Rate in $= 5\frac{\text{gal}}{\text{min}} \times \frac{1}{2}\frac{\text{lb}}{\text{gal}} = 2.5\frac{\text{lb}}{\text{min}}$
- Rate out $= 10\frac{\text{gal}}{\text{min}} \times \frac{S}{V}\frac{\text{lb}}{\text{gal}}$
  - $V(t) = 500 - 5t.$
  - Rate out $= \frac{10S}{500-5t}\frac{\text{lb}}{\text{min}}$

$$\frac{dS}{dt} = \text{Rate in - Rate out}$$
$$= 2.5 - \frac{2S}{100 - t},$$

- General solution:

$$S(t) = 2.5(100 - t) + C(100 - t)^2.$$

- Particular solution:

$$S(t) = 2.5(100 - t) - \frac{(100 - t)^2}{50}.$$