Math 211

Lecture #2

Solutions to Differential Equations

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Differential Equations

An equation involving an unknown function and one or more of its derivatives, in addition to the independent variable.

- Example: $y' = 2ty$
- General equation: $y' = f(t, y)$
- t is the *independent variable*.
- $y = y(t)$ is the *unknown function*.
- $y' = 2ty$ is of *order 1.*
- $y'' + 3yy' = \cos t$ is *second order*.

Equations and Solutions

$$y' = f(t, y) \qquad y' = 2ty$$

A *solution* is a function $y(t)$, defined for t in an interval, which is differentiable at each point and satisfies

$$y'(t) = f(t, y(t))$$

for every point t in the interval.

- What is a function?
- An ODE is a function generator.

Example: $y' = 2ty$

Claim: $y(t) = e^{t^2}$ is a solution.

- Verify by substitution.
  - Left-hand side: $y'(t) = 2te^{t^2}$
  - Right-hand side: $2ty(t) = 2te^{t^2}$
- Therefore $y'(t) = 2ty(t)$, if $y(t) = e^{t^2}$.
- Verification by substitution is always available.

Is $y(t) = e^t$ a solution to the equation $y' = 2ty$?

- Check by substitution.
  - Left-hand side: $y'(t) = e^t$
  - Right-hand side: $2ty(t) = 2te^t$
- Therefore $y'(t) \neq 2ty(t)$, if $y(t) = e^t$.
- $y(t) = e^t$ is *not* a solution to the equation $y' = 2ty$.

Types of Solutions

For the equation $y' = 2ty$

- $y(t) = \frac{1}{2}e^{t^2}$ is a solution. It is a *particular solution.*
- $y(t) = Ce^{t^2}$ is a solution for any constant C. This is a *general solution.*

General solutions contain arbitrary constants. Particular solutions do not.

Initial Value Problem (IVP)

A differential equation & an initial condition.

- Example: $y' = -2ty$ with $y(0) = 4$.
- General solution: $y(t) = Ce^{-t^2}$.
- Plug in the initial condition:

$$y(0) = 4,$$
$$Ce^0 = 4,$$
$$C = 4$$

Solution to the IVP: $y(t) = 4e^{-t^2}$.

An ODE is a Function Generator

Example: $y' = y^2 - t$, $y(0) = 0$

- There is no solution to this IVP which can be given using a formula.
- Nevertheless, there is a solution. We can find as many terms in the power series for $y(t)$ as we want.

$$y(t) = -\frac{1}{2}t^2 + \frac{1}{20}t^5 - \frac{1}{160}t^8 + \dots$$

Normal Form of an Equation

$$y' = f(t, y)$$

Example: $(1 + t^2)y' + y^2 = t^3$

- This equation is not in normal form.
- Solve for y' to put the equation into normal form:

$$y' = \frac{t^3 - y^2}{1 + t^2}$$

Interval of Existence

The largest interval over which a solution can exist.

- Example: $y' = 1 + y^2 \quad$ with $\quad y(0) = 1$
  - General solution: $y(t) = \tan(t + C)$
  - Initial Condition: $y(0) = 1 \Leftrightarrow C = \pi/4.$
- Solution: $y(t) = \tan(t + \pi/4)$ exists and is continuous for

$$-\pi/2 < t + \pi/4 < \pi/2$$

or for

$$-3\pi/4 < t < \pi/4.$$

Geometric Interpretation of

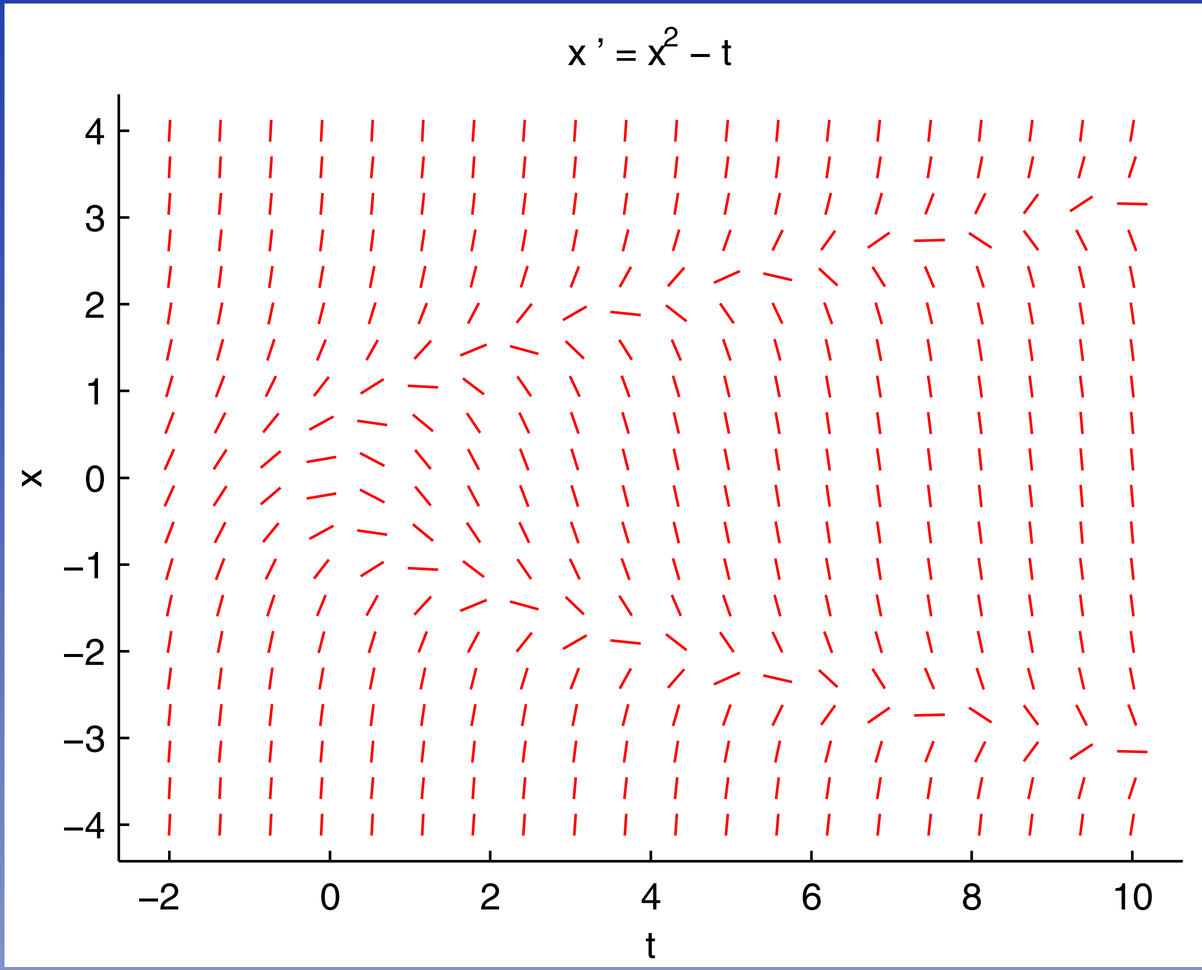
$$y' = f(t, y)$$

If $y(t)$ is a solution, and $y(t_0) = y_0$, then

$$y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0).$$

- The slope to the graph of $y(t)$ at the point (t_0, y_0) is given by $f(t_0, y_0)$.
- Imagine a small line segment attached to each point of the (t, y) plane with the slope $f(t, y)$.

The Direction Field



Autonomous Equations

General equation:

$$\frac{dy}{dt} = f(t, y) \qquad \frac{dy}{dt} = t - y^2$$

Autonomous equation:

$$\frac{dy}{dt} = f(y) \qquad \frac{dy}{dt} = y(1 - y)$$

In an *autonomous equation* the right-hand side has no explicit dependence on the independent variable.

Equilibrium Points

Autonomous equation:

$$\frac{dy}{dt} = f(y) \qquad \frac{dy}{dt} = y(2-y)/3$$

- *Equilibrium point:*

$$f(y_0) = 0 \quad y_0 = 0 \quad \text{or} \quad 2$$

- *Equilibrium solution:*

$$y(t) = y_0 \quad y(t) = 0 \quad \text{and} \quad y(t) = 2$$

Between Equilibrium Points

- $\frac{dy}{dt} = f(y) > 0 \Rightarrow \; y(t)$ is increasing.
- $\frac{dy}{dt} = f(y) < 0 \Rightarrow \; y(t)$ is decreasing.

Example: $\frac{dy}{dt} = y(2 - y)/3$

Separable Equations

General differential equation:

$$\frac{dy}{dt} = f(t, y) \qquad \frac{dy}{dt} = t - y^2$$

Separable differential equation:

$$\frac{dy}{dt} = g(y)h(t) \qquad \frac{dy}{dt} = t \sec y$$

In a *separable equation* the right-hand side is a product of a function of the independent variable (t) and a function of the unknown function (y).

- Autonomous equations are separable.

Solving Separable Equations

$$\frac{dy}{dt} = t \sec y$$

- Step 1: Separate the variables:

$$\frac{dy}{\sec y} = t\,dt \quad \text{or} \quad \cos y\,dy = t\,dt$$

We have to worry about dividing by 0, but in this case $\sec y$ is never equal to 0.

Step 2: Integrate both sides

$$\int \cos y \, dy = \int t \, dt$$

$$\sin(y) + C_1 = \frac{1}{2}t^2 + C_2 \quad \text{or}$$

$$\sin(y) = \frac{1}{2}t^2 + C$$

where $C = C_2 - C_1$.

Step 3: Solve for $y(t)$

$$\sin(y) = \frac{1}{2}t^2 + C$$

$$y(t) = \arcsin\left(C + \frac{1}{2}t^2\right).$$

This is the general solution to $\frac{dy}{dt} = t \sec y$.

Solving Separable Equations

$$\frac{dy}{dt} = g(y)h(t)$$

The three step solution process:

1. Separate the variables. $\dfrac{dy}{g(y)} = h(t)\,dt$
2. Integrate both sides. $\displaystyle\int \frac{dy}{g(y)} = \int h(t)\,dt$
3. Solve for $y(t)$.

Return

Examples

- $y' = ry$
- $R' = \dfrac{\sin t}{1+R}$ with $R(0) = 1, -2, -1$
- $x' = \dfrac{3t^2 x}{1+2x^2}$ with $x(0) = 1, 0$
- $y' = 1 + y^2$ with $y(0) = -1, 0, 1$