

Math 211

Lecture #3
Separable Equations

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Autonomous Equations

General equation:

$$\frac{dy}{dt} = f(t, y) \quad \frac{dy}{dt} = t - y^2$$

Autonomous equation:

$$\frac{dy}{dt} = f(y) \quad \frac{dy}{dt} = y(2 - y)/3$$

In an *autonomous equation* the right-hand side has no explicit dependence on the independent variable.

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Equilibrium Points

Autonomous equation:

$$\frac{dy}{dt} = f(y) \quad \frac{dy}{dt} = y(2 - y)/3$$

- *Equilibrium point:*

$$f(y_0) = 0 \quad y_0 = 0 \quad \text{or} \quad 2$$

- *Equilibrium solution:*

$$y(t) = y_0 \quad y(t) = 0 \quad \text{and} \quad y(t) = 2$$

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Between Equilibrium Points

- $\frac{dy}{dt} = f(y) > 0 \Rightarrow y(t)$ is increasing.
- $\frac{dy}{dt} = f(y) < 0 \Rightarrow y(t)$ is decreasing.

Example: $\frac{dy}{dt} = y(2 - y)/3$

Equilibrium point

Separable Equations

General differential equation:

$$\frac{dy}{dt} = f(t, y) \quad \frac{dy}{dt} = t - y^2$$

Separable differential equation:

$$\frac{dy}{dt} = g(y)h(t) \quad \frac{dy}{dt} = t \sec y$$

In a *separable equation* the right-hand side is a product of a function of the independent variable (t) and a function of the unknown function (y).

- Autonomous equations are separable.

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Solving Separable Equations

$$\frac{dy}{dt} = t \sec y$$

- Step 1: Separate the variables:

$$\frac{dy}{\sec y} = t dt \quad \text{or} \quad \cos y dy = t dt$$

We have to worry about dividing by 0, but in this case $\sec y$ is never equal to 0.

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Separable

Step 2: Integrate both sides

$$\int \cos y \, dy = \int t \, dt$$

$$\sin(y) + C_1 = \frac{1}{2}t^2 + C_2 \quad \text{or}$$

$$\sin(y) = \frac{1}{2}t^2 + C$$

where $C = C_2 - C_1$.

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[Step 1](#)

Step 3: Solve for $y(t)$

$$\sin(y) = \frac{1}{2}t^2 + C$$

$$y(t) = \arcsin\left(C + \frac{1}{2}t^2\right).$$

This is the general solution to $\frac{dy}{dt} = t \sec y$.

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[Step 1](#)

[Step 2](#)

Solving Separable Equations

$$\frac{dy}{dt} = g(y)h(t)$$

The three step solution process:

1. Separate the variables. $\frac{dy}{g(y)} = h(t) \, dt$
2. Integrate both sides. $\int \frac{dy}{g(y)} = \int h(t) \, dt$
3. Solve for $y(t)$.

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Examples

- $y' = ry$
- $y' = 2ty$
- $R' = \frac{\sin t}{1+R}$ with $R(0) = 1, -2, -1$
- $x' = \frac{3t^2x}{1+2x^2}$ with $x(0) = 1, 0$
- $y' = 1 + y^2$ with $y(0) = -1, 0, 1$

[Solution procedure](#)

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Why the Method Works

$$\begin{aligned} \frac{dy}{dt} &= g(y)h(t) \\ \frac{1}{g(y)} \frac{dy}{dt} &= h(t) \quad \text{if } g(y) \neq 0 \\ \int \frac{1}{g(y)} \frac{dy}{dt} dt &= \int h(t) dt \\ \int \frac{1}{g(y)} dy &= \int h(t) dt \end{aligned}$$

[Solution procedure](#)

[Examples](#)

Models of Motion

History of models of planetary motion

- Babylonians - 3000 years ago
 - ♦ Initiated the systematic study of astronomy.

Greeks

- Descriptive model
 - ♦ Geocentric model
 - ♦ Epicycles
- Enabled predictions
- No causal explanation

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Nicholas Copernicus (1543)

- Shifted the center of the universe to the sun.
- Less epicycles required.
- Still descriptive and not causal.
- Major change in human understanding of their place in the universe.

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[Greeks](#)

Johann Kepler (1609)

- Based on experimental work of Tycho Brahe.
- Ellipses instead of epicycles.
 - ♦ Sun at a focus of the ellipse.
- Three laws of planetary motion.
- Still descriptive and not causal.

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[Greeks](#)

[Copernicus](#)

Isaac Newton

- Three major contributions.
 - ♦ Fundamental theorem of calculus.
 - Invention of calculus.
 - ♦ Laws of mechanics.
 - Second law — $F = ma$.
 - ♦ Universal law of gravity.
 - ♦ *Principia Mathematica* 1687

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Isaac Newton

- Laws of mechanics and gravitation were based on his own experiments and his understanding of the experiments of others.
- Derived Kepler's three laws of planetary motion.
- Causal explanation.
 - ♦ For any mechanical motion.

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[Greeks](#)

[Copernicus](#)

[Kepler](#)

[Newton 1](#)

Isaac Newton

- *The Life of Isaac Newton* by Richard Westfall, Cambridge University Press 1993.
- Problems
 - ♦ Force of gravity was action at a distance.
 - ♦ Physical anomalies.

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Albert Einstein

- Special theory of relativity – 1905.
- General theory of relativity – 1916.
 - ♦ Gravity is due to curvature of space-time.
 - ♦ Curvature is caused by mass.
 - ♦ Explains action at a distance.
- All known anomalies explained.

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[Newton Problems](#)

Unified Theories

- Four fundamental forces.
 - ♦ Gravity, electromagnetism, strong nuclear, and weak nuclear.
- Last three unified by quantum mechanics.
 - ♦ Quantum chromodynamics.
- Attempts to include gravity.
 - ♦ String theory.

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Unified Theories

- String theory.
 - ♦ *The elegant universe : superstrings, hidden dimensions, and the quest for the ultimate theory* by Brian Greene, W.W.Norton, New York 1999.

[Unified theories](#)

The Modeling Process

- It is based on experiment and/or observation.
- It is iterative.
 - ♦ For motion we have ≥ 6 iterations.
 - ♦ After each change in the model it must be checked by experimentation and observation.
- It is rare that a model captures all aspects of the phenomenon.