

# Math 211

Lecture #3

Separable Equations

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# Autonomous Equations

General equation:

$$\frac{dy}{dt} = f(t, y) \quad \frac{dy}{dt} = t - y^2$$

Autonomous equation:

$$\frac{dy}{dt} = f(y) \quad \frac{dy}{dt} = y(2 - y)/3$$

In an *autonomous equation* the right-hand side has no explicit dependence on the independent variable.

# Equilibrium Points

Autonomous equation:

$$\frac{dy}{dt} = f(y) \quad \frac{dy}{dt} = y(2 - y)/3$$

- *Equilibrium point:*

$$f(y_0) = 0 \quad y_0 = 0 \quad \text{or} \quad 2$$

- *Equilibrium solution:*

$$y(t) = y_0 \quad y(t) = 0 \quad \text{and} \quad y(t) = 2$$

## Between Equilibrium Points

- $\frac{dy}{dt} = f(y) > 0 \Rightarrow y(t)$  is increasing.
- $\frac{dy}{dt} = f(y) < 0 \Rightarrow y(t)$  is decreasing.

Example:  $\frac{dy}{dt} = y(2 - y)/3$

# Separable Equations

General differential equation:

$$\frac{dy}{dt} = f(t, y) \quad \frac{dy}{dt} = t - y^2$$

Separable differential equation:

$$\frac{dy}{dt} = g(y)h(t) \quad \frac{dy}{dt} = t \sec y$$

In a *separable equation* the right-hand side is a product of a function of the independent variable ( $t$ ) and a function of the unknown function ( $y$ ).

- **Autonomous equations** are separable.

## Solving Separable Equations

$$\frac{dy}{dt} = t \sec y$$

- Step 1: Separate the variables:

$$\frac{dy}{\sec y} = t dt \quad \text{or} \quad \cos y dy = t dt$$

We have to worry about dividing by 0, but in this case  $\sec y$  is never equal to 0.

## Step 2: Integrate both sides

$$\int \cos y \, dy = \int t \, dt$$
$$\sin(y) + C_1 = \frac{1}{2}t^2 + C_2 \quad \text{or}$$
$$\sin(y) = \frac{1}{2}t^2 + C$$

where  $C = C_2 - C_1$ .

### Step 3: Solve for $y(t)$

$$\sin(y) = \frac{1}{2}t^2 + C$$

$$y(t) = \arcsin\left(C + \frac{1}{2}t^2\right).$$

This is the general solution to  $\frac{dy}{dt} = t \sec y$ .

# Solving Separable Equations

$$\frac{dy}{dt} = g(y)h(t)$$

The three step solution process:

1. **Separate** the variables.  $\frac{dy}{g(y)} = h(t) dt$
2. **Integrate** both sides.  $\int \frac{dy}{g(y)} = \int h(t) dt$
3. **Solve** for  $y(t)$ .

## Examples

- $y' = ry$
- $y' = 2ty$
- $R' = \frac{\sin t}{1 + R}$  with  $R(0) = 1, -2, -1$
- $x' = \frac{3t^2 x}{1 + 2x^2}$  with  $x(0) = 1, 0$
- $y' = 1 + y^2$  with  $y(0) = -1, 0, 1$

## Why the Method Works

$$\frac{dy}{dt} = g(y)h(t)$$

$$\frac{1}{g(y)} \frac{dy}{dt} = h(t) \quad \text{if } g(y) \neq 0$$

$$\int \frac{1}{g(y)} \frac{dy}{dt} dt = \int h(t) dt$$

$$\int \frac{1}{g(y)} dy = \int h(t) dt$$

# Models of Motion

History of models of planetary motion

- Babylonians - 3000 years ago
  - ◆ Initiated the systematic study of astronomy.

# Greeks

- Descriptive model
  - ◆ Geocentric model
  - ◆ Epicycles
- Enabled predictions
- No causal explanation

## Nicholas Copernicus (1543)

- Shifted the center of the universe to the sun.
- Less epicycles required.
- Still descriptive and not causal.
- Major change in human understanding of their place in the universe.

## Johann Kepler (1609)

- Based on experimental work of Tycho Brahe.
- Ellipses instead of epicycles.
  - ◆ Sun at a focus of the ellipse.
- Three laws of planetary motion.
- Still descriptive and not causal.

# Isaac Newton

- Three major contributions.
  - ◆ Fundamental theorem of calculus.
    - ▶ Invention of calculus.
  - ◆ Laws of mechanics.
    - ▶ Second law —  $F = ma$ .
  - ◆ Universal law of gravity.
  - ◆ *Principia Mathematica* 1687

# Isaac Newton

- Laws of mechanics and gravitation were based on his own experiments and his understanding of the experiments of others.
- Derived Kepler's three laws of planetary motion.
- Causal explanation.
  - ◆ For any mechanical motion.

# Isaac Newton

- *The Life of Isaac Newton* by Richard Westfall, Cambridge University Press 1993.
- Problems
  - ◆ Force of gravity was action at a distance.
  - ◆ Physical anomalies.

# Albert Einstein

- Special theory of relativity – 1905.
- General theory of relativity – 1916.
  - ◆ Gravity is due to curvature of space-time.
  - ◆ Curvature is caused by mass.
  - ◆ Explains action at a distance.
- All known anomalies explained.

# Unified Theories

- Four fundamental forces.
  - ◆ Gravity, electromagnetism, strong nuclear, and weak nuclear.
- Last three unified by quantum mechanics.
  - ◆ Quantum chromodynamics.
- Attempts to include gravity.
  - ◆ String theory.

# Unified Theories

- String theory.
  - ◆ *The elegant universe : superstrings, hidden dimensions, and the quest for the ultimate theory* by Brian Greene, W.W.Norton, New York 1999.

# The Modeling Process

- It is based on experiment and/or observation.
- It is iterative.
  - ◆ For motion we have  $\geq 6$  iterations.
  - ◆ After each change in the model it must be checked by experimentation and observation.
- It is rare that a model captures all aspects of the phenomenon.