Math 211

Lecture #5

Linear Equations

September 7, 2001

Air Resistance

Acts in the direction opposite to the velocity. Therefore

$$R(x, v) = -r(x, v)v \quad \text{where} \quad r(x, v) \geq 0.$$

There are many models. We will look at two different cases.

1. The resistance is proportional to velocity.
2. The magnitude of the resistance is proportional to the square of the velocity.

Return

Magnitude of Resistance Proportional to the Square of the Velocity

- $R(x, v) = -k|v|v$, k a positive constant.
- Total force: $F = -mg - k|v|v$.
- Equation of motion:

$$mx'' = -mg - k|v|v \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -\frac{mg + k|v|v}{m}. \end{aligned}$$

- The equation for v is separable.

Return Resistance

- Suppose a ball is dropped from a high point. Then $v < 0$.
- The equation is $v' = \dfrac{-mg + kv^2}{m}$.
- The solution is

$$v(t) = \sqrt{\frac{mg}{k}}\,\frac{Ae^{-2t\sqrt{kg/m}} - 1}{Ae^{-2t\sqrt{kg/m}} + 1}.$$

- The terminal velocity is

$$v_{\mathrm{term}} = -\sqrt{mg/k}.$$

Solving for $x(t)$

- Integrating $x' = v(t)$ is sometimes hard.
- Use the trick (see Exercise 2.3.7):

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

- The equation

$$v\frac{dv}{dx} = a$$

  is usually separable.

Problem

A ball is projected from the surface of the earth with velocity v_0. How high does it go?

- At $t = 0$, we have $x(0) = 0$ and $v(0) = v_0$.
- At the top we have $t = T$, $x(T) = x_{\max}$, and $v(T) = 0$.

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$$R = -k|v|v$$

Since $v > 0$, the acceleration is $a = -\dfrac{mg + kv^2}{m}$. The equation $v\dfrac{dv}{dx} = a$ becomes

$$\int_{v_0}^{0} \frac{v\,dv}{kv^2 + mg} = -\int_0^{x_{\max}} \frac{dx}{m}.$$

Solving, we get

$$x_{\max} = \frac{m}{2k} \ln\left(1 + \frac{kv_0^2}{mg}\right).$$

Problem

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Linear Equations

A *linear equation* is one of the form

$$x' = a(t)x + f(t).$$

- Example: $x' = \tan(t)x + 3\sin^2(t)$
- The unknown function x and its derivative must appear *linearly*.
- The equation is *homogeneous* if $f = 0$
  - $x' = a(t)x$, e.g. $x' = \tan(t)x$
- The equation is *inhomogeneous* if $f \neq 0$

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Homogenous Linear Equations

- Homogeneous linear equations are separable.

$$\frac{dx}{dt} = a(t)x$$

$$\int \frac{dx}{x} = \int a(t)\,dt$$

$$x(t) = Ae^{\int a(t)\,dt}$$

- Example: $x' = \tan(t)x$.

$$x(t) = A\sec t$$

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Example: $x' = \tan(t)x + 3\sin^2(t)$

- Rewrite as $x' - \tan(t)x = 3\sin^2(t)$
- Multiply by $\cos t$.

$$\cos(t)x' - \sin(t)x = 3\sin^2(t)\cos(t)$$

The left hand side is the derivative of $\cos(t)x$. So

$$[\cos(t)x]' = 3\sin^2(t)\cos(t)$$

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- Integrate

$$\cos(t)x(t) = 3\int \sin^2(t)\cos(t)\,dt = \sin^3(t) + C$$

- Solve for x

$$x(t) = \frac{\sin^3(t) + C}{\cos(t)}.$$

How did we do that? Can we do it in general?

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Solution pt. 1

The Key Step for $x' = ax + f$

- Rewrite as $x' - ax = f$.
- Multiply by a function $u(t)$ so that

$$u[x' - ax] = [ux]'$$
$$ux' - aux = ux' + u'x$$

  - True if $u' = -au$. Linear, homogeneous

    $$u(t) = e^{-\int a(t)\,dt} \quad \text{is one solution.}$$

  - u is called an *integrating factor*.

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Solution pt. 1

Solving the Linear Equation

$$x' = a(t)x + f(t)$$

Four step process:

1. Rewrite as $x' - ax = f$.
2. Multiply by the integrating factor

$$u(t) = e^{-\int a(t)\,dt}.$$

   Equation becomes $[ux]' = ux' - aux = uf$.

3. Integrate: $u(t)x(t) = \int u(t)f(t)\,dt + C$.
4. Solve for $x(t)$.

Return

Examples

- $x' = -4x + 8, \quad x(0) = 0.$
- $x' = 2tx + e^{t^2}, \quad x(0) = 1.$
- $y' = 3y - t, \quad y(0) = 2.$
- $z' = (z + 1)\cos t, \quad z(\pi) = -1.$

Solution method