Math 211

Lecture #6

Mixing Problems

September 10, 2001

Solving the Linear Equation

$$x' = a(t)x + f(t)$$

Four step process:

1. Rewrite as $x' - ax = f$.
2. Multiply by the integrating factor

$$u(t) = e^{-\int a(t)\,dt}.$$

   Equation becomes $[ux]' = ux' - aux = uf$.
3. Integrate: $u(t)x(t) = \int u(t)f(t)\,dt + C$.
4. Solve for $x(t)$.

Mixing Problem #1

A tank originally holds 500 gallons of pure water. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 5 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.

Model

- $S(t) =$ the amount of sugar in the tank in lbs.
- *Concentration* $=$ pounds per unit volume.
  - $c(t) = \dfrac{S(t)}{V} \dfrac{\text{lbs}}{\text{gal}}.$
- Modeling is easier in terms of the total amount, $S(t)$.
- Draw a picture.
- We must compute the rate of change of S in two ways.
  - The mathematical way: rate of change $= dS/dt$.
  - The application way.
    - This where the real modeling takes place.

The Rate of Change of $S(t)$

- Balance Law:

  Rate of change = Rate in - Rate out

- Rate = volume rate × concentration
- For the problem
  - Rate in $= 5\,\frac{\text{gal}}{\text{min}} \times \frac{1}{2}\frac{\text{lb}}{\text{gal}} = 2.5\,\frac{\text{lb}}{\text{min}}$
  - Rate out $= 5\,\frac{\text{gal}}{\text{min}} \times \frac{S}{500}\frac{\text{lb}}{\text{gal}} = \frac{S}{100}\frac{\text{lb}}{\text{min}}$

Solution

$$\begin{aligned}\frac{dS}{dt} &= \text{Rate in - Rate out} \\ &= 2.5 - \frac{S}{100}.\end{aligned}$$

- General solution: $S(t) = 250 + Ce^{-t/100}$.
- Particular solution: $S(t) = 250(1 - e^{-t/100})$.

Other possible initial conditions

- There is initially 20 lbs of sugar in the tank.
- The concentration of sugar in the tank at $t = 0$ is 1 lb/gallon.

Mixing Problem #2

A tank originally holds 500 gallons of sugar water with a concentration of $\frac{1}{10}$ lb/gal. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 10 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.

Solution

- Rate in $= 5 \frac{\text{gal}}{\text{min}} \times \frac{1}{2} \frac{\text{lb}}{\text{gal}} = 2.5 \frac{\text{lb}}{\text{min}}$
- Rate out $= 10 \frac{\text{gal}}{\text{min}} \times \frac{S}{V} \frac{\text{lb}}{\text{gal}}$
  - $V(t) = 500 - 5t.$
  - Rate out $= \frac{10S}{500 - 5t} \frac{\text{lb}}{\text{min}}$

$$\frac{dS}{dt} = \text{Rate in - Rate out}$$
$$= 2.5 - \frac{2S}{100 - t},$$

- General solution:

$$S(t) = 2.5(100 - t) + C(100 - t)^2.$$

- Particular solution:

$$S(t) = 2.5(100 - t) - \frac{(100 - t)^2}{50}.$$

Qualitative Analysis

- Do solutions always exist?
- How many solutions are there to an ODE?
  - How many solutions are there to an initial value problem?
- Can we predict the behavior of solutions without having a formula?

Example of Non-existence

- Initial value problem:

$$\sin(t)y' = \cos(t)y + \sin^2(t) \quad \text{with} \quad y(0) = 1.$$

- Every solution to the differential equation has the form

$$y(t) = t\sin t + C\sin t.$$

- Hence $y(0) = 0$ for every solution. The IVP with $y(0) = 1$ has *no solution.*

Existence of Solutions

- Put the equation $\sin(t)y' = \cos(t)y + \sin^2(t)$ into normal form
$$y' = \frac{\cos t}{\sin t}y + \sin t.$$
- The RHS is not defined at $t = 0$.
- If we require the RHS to be continuous there is always a solution to an initial value problem.

Existence Theorem

Theorem: Suppose the function $f(t, y)$ is defined and continuous in the rectangle R in the ty-plane. Then given any point $(t_0, y_0) \in R$, the initial value problem

$$y' = f(t, y) \quad \text{with} \quad y(t_0) = y_0$$

has a solution $y(t)$ defined in an interval containing t_0. Furthermore the solution will be defined at least until the solution curve $t \to (t, y(t))$ leaves the rectangle R.

What is a Theorem?

- A theorem is a logical statement.
- It contains
  - *hypotheses* (the assumptions made)
  - and *conclusions*
- The conclusions are guaranteed to be true if the hypotheses are true.
- The implication goes only one way.

Example of a "Theorem"

If it rains the sidewalks get wet.

- Hypothesis — *If it rains*
- Conclusion — *the sidewalks get wet*

Mathematics and Proof

- Theorems are proved by logical deduction.
- All of mathematics comes from a small number of very basic assumptions.
  - Called *axioms* or *postulates*.
- True of all parts of mathematics.
  - An algebraic derivation is an example of a proof.
- Definitions are not theorems.

Existence of a Solution

- The existence theorem does not guarantee an explicitly defined solution.
- In the proof, the solution is constructed as the limit of a sequence of explicitly defined functions.
- Frequently no explicit formula is possible.
- An ordinary differential equation is a function generator.