

Math 211

Lecture #8

Qualitative Analysis

September 14, 2001

Qualitative Analysis

- Ways to discover the properties of solutions without solving the equation.
- Works best with autonomous equations

$$y' = f(y)$$

- Example: $y' = \sin y$

Properties of Autonomous Equations

- The direction field does not depend on t
- Solution curves can be translated left and right to get other solution curves.
 - ◆ If $y(t)$ is a solution, so is $y_1 = y(t + c)$ for any constant c .

Equilibrium Points & Solutions

Autonomous equation: $y' = f(y)$.

- Equilibrium point: $f(y_0) = 0$.
- Equilibrium solution: $y(t) = y_0$.
- Example: $y' = \sin y$
 - ♦ $\sin y = 0 \Leftrightarrow y = k\pi, \quad k = 0, \pm 1, \dots$
 - ♦ $y' = \sin y$ has infinitely many equilibrium solutions:
 - $y_k(t) = k\pi \quad \text{for } k = 0, \pm 1, \pm 2, \dots$

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Between the Equilibrium Points

$$\begin{aligned} 0 < y < \pi &\Rightarrow \sin y > 0 \\ &\Rightarrow y'(t) = \sin y(t) > 0 \\ &\Rightarrow y(t) \text{ is increasing} \end{aligned}$$

- By uniqueness, $0 < y(t) < \pi$ for all t .
- Thus $y(t) \nearrow \pi$ as $t \rightarrow \infty$
and $y(t) \searrow 0$ as $t \rightarrow -\infty$

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Eq. Pt.

Between the Equilibrium Points

$$\begin{aligned} -\pi < y < 0 &\Rightarrow \sin y < 0 \\ &\Rightarrow y'(t) = \sin y(t) < 0 \\ &\Rightarrow y(t) \text{ is decreasing} \end{aligned}$$

- By uniqueness, $0 > y(t) > -\pi$ for all t .
- Thus $y(t) \searrow -\pi$ as $t \rightarrow \infty$
and $y(t) \nearrow 0$ as $t \rightarrow -\infty$

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Eq. Pt.

$f(y) > 0$

Stable & Unstable EPs

An equilibrium point y_0 is

- *asymptotically stable* if all solutions starting near y_0 converge to y_0 as $t \rightarrow \infty$.
- *unstable* if there are solutions starting arbitrarily close to y_0 which move away from y_0 as t increases.
- There are 4 possibilities:

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The Phase Line for $y' = f(y)$

- The phase line is a y -axis, showing
 - ♦ the equilibrium points and
 - ♦ the direction of the flow between the equilibrium points.
- The y -axis in the plot of $y \rightarrow f(y)$.
- The y -axis in the ty -plane where solutions are plotted.

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Terminal Velocity

- Magnitude of the resistance proportional to the square of the velocity:

$$v' = -g - k|v|v/m$$

- One equilibrium point at

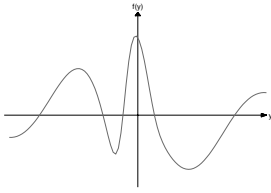
$$v_{\text{term}} = -\sqrt{\frac{mg}{k}}.$$

- v_{term} is asymptotically stable.

Type

Qualitative Analysis of $y' = f(y)$.

1. Graph $y \rightarrow f(y)$.

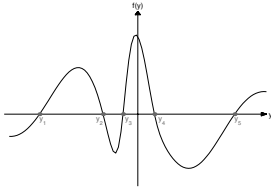


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7 steps

Qualitative Analysis of $y' = f(y)$.

2. Find the equilibrium points where $f(y) = 0$.



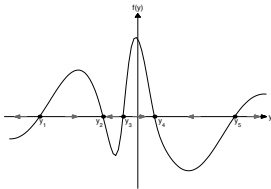
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Graph

7 steps

Qualitative Analysis of $y' = f(y)$.

3. Determine the behavior between eq. pts.



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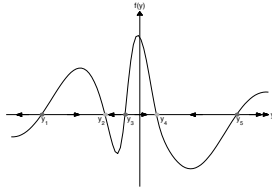
Graph

EPs

7 steps

Qualitative Analysis of $y' = f(y)$.

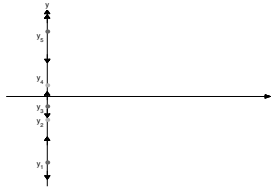
4. Analyze the equilibrium points.



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Qualitative Analysis of $y' = f(y)$.

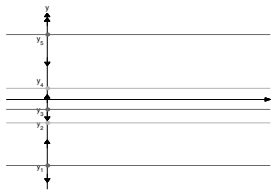
5. Transfer the phase line to ty -space.



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Qualitative Analysis of $y' = f(y)$.

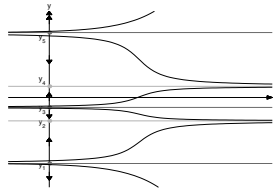
6. Plot the equilibrium solutions.



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Qualitative Analysis of $y' = f(y)$.

7. Plot other solutions approximately.



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Seven Steps

1. Graph $y \rightarrow f(y)$.
2. Find the equilibrium points where $f(y) = 0$.
3. Determine the behavior between eq. pts.
4. Analyze the equilibrium points.
5. Transfer the phase line to ty -space.
6. Plot the equilibrium solutions.
7. Plot other solutions approximately.