Math 211

Lecture #8

Qualtitative Analysis

September 14, 2001

Qualitative Analysis

- Ways to discover the properties of solutions without solving the equation.
- Works best with autonomous equations

$$y' = f(y)$$

- Example: $y' = \sin y$

Properties of Autonomous Equations

- The direction field does not depend on t
- Solution curves can be translated left and right to get other solution curves.
  - If $y(t)$ is a solution, so is $y_1 = y(t + c)$ for any constant c.

Equilibrium Points & Solutions

Autonomous equation: $y' = f(y)$.

- Equilibrium point: $f(y_0) = 0$.
- Equilibrium solution: $y(t) = y_0$.
- Example: $y' = \sin y$
  - $\sin y = 0 \quad \Leftrightarrow \quad y = k\pi, \quad k = 0, \pm 1, \ldots$
  - $y' = \sin y$ has infinitely many equilibrium solutions:
    - $y_k(t) = k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$

Return

Between the Equilibrium Points

$$
\begin{aligned}
0 < y < \pi \ &\Rightarrow\ \sin y > 0 \\
&\Rightarrow\ y'(t) = \sin y(t) > 0 \\
&\Rightarrow\ y(t) \text{ is increasing}
\end{aligned}
$$

- By uniqueness, $0 < y(t) < \pi$ for all t.
- Thus $y(t) \nearrow \pi$ as $t \longrightarrow \infty$
  and $y(t) \searrow 0$ as $t \longrightarrow -\infty$

Between the Equilibrium Points

$$\begin{aligned} -\pi < y < 0 \;&\Rightarrow\; \sin y < 0 \\ &\Rightarrow\; y'(t) = \sin y(t) < 0 \\ &\Rightarrow\; y(t) \text{ is decreasing} \end{aligned}$$

- By uniqueness, $0 > y(t) > -\pi$ for all t.
- Thus $y(t) \searrow -\pi$ as $t \to \infty$
  and $y(t) \nearrow 0$ as $t \to -\infty$

Stable & Unstable EPs

An equilibrium point y_0 is

- *asymptotically stable* if all solutions starting near y_0 converge to y_0 as $t \longrightarrow \infty$.
- *unstable* if there are solutions starting arbitrarily close to y_0 which move away from y_0 as t increases.
- There are 4 possibilities:

The Phase Line for $y' = f(y)$

- The phase line is a y-axis, showing
  - the equilibrium points and
  - the direction of the flow between the equilibrium points.
- The y-axis in the plot of $y \to f(y)$.
- The y-axis in the ty-plane where solutions are plotted.

Terminal Velocity

- Magnitude of the resistance proportional to the square of the velocity:

$$v' = -g - k|v|v/m$$

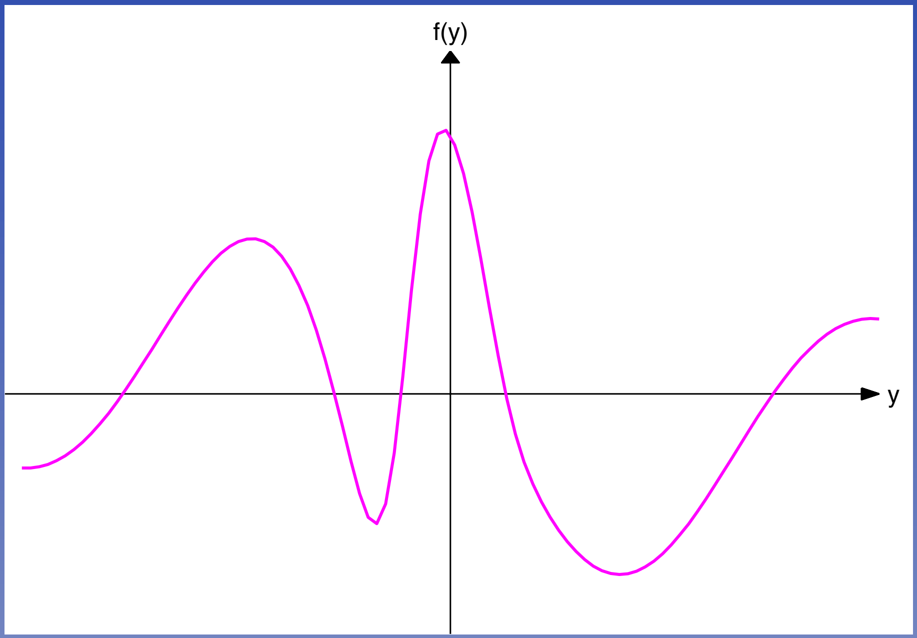
- One equilibrium point at

$$v_{\text{term}} = -\sqrt{\frac{mg}{k}}.$$

- v_{term} is asymptotically stable.

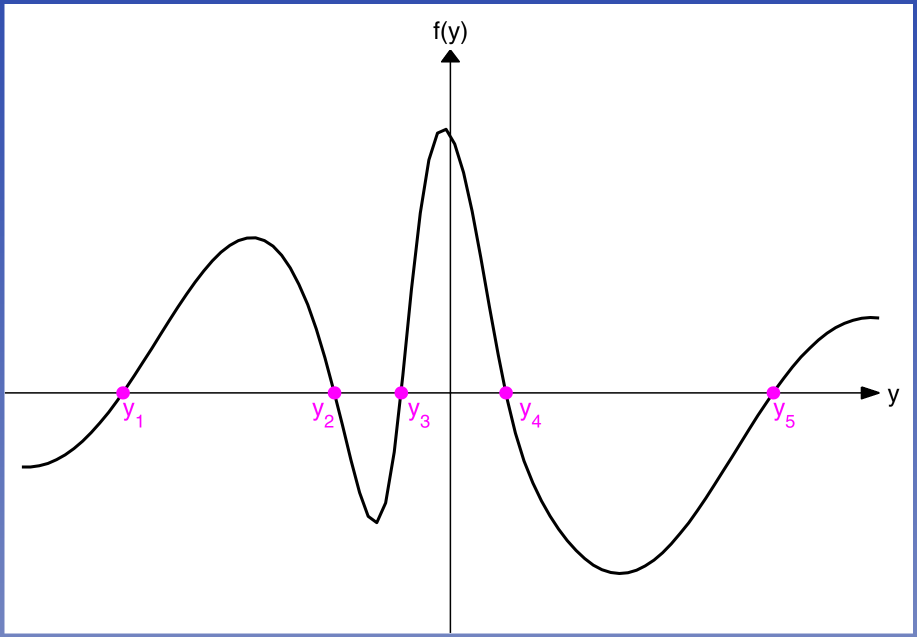
Qualitative Analysis of $y' = f(y)$.

1. Graph $y \to f(y)$.



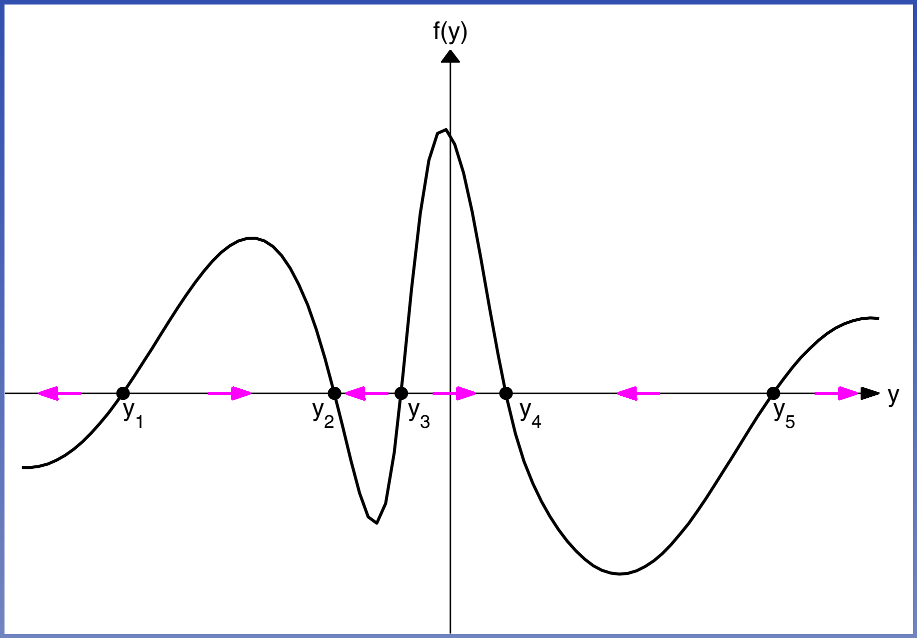
Qualitative Analysis of $y' = f(y)$.

2. Find the equilibrium points where $f(y) = 0$.



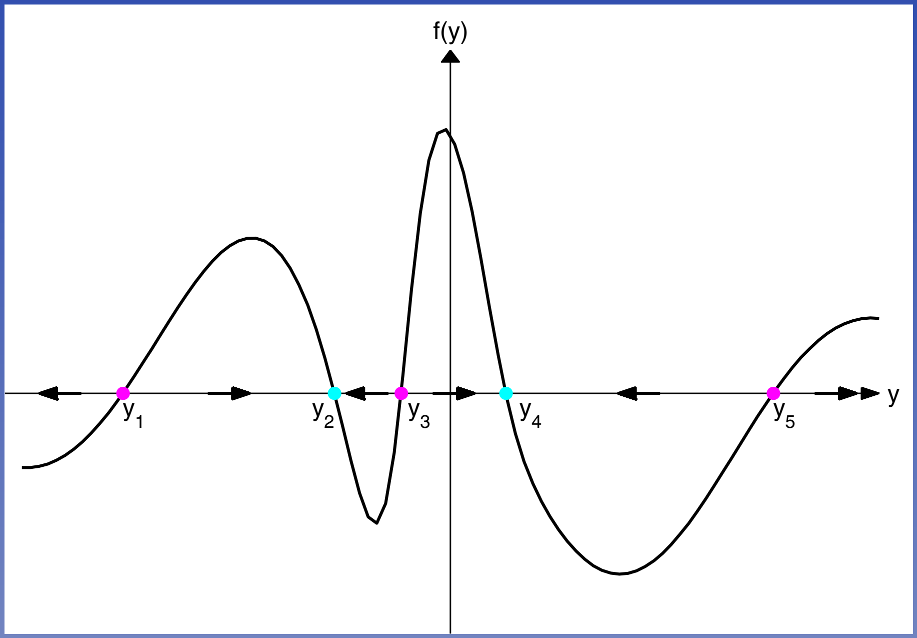
Qualitative Analysis of $y' = f(y)$.

3. Determine the behavior between eq. pts.



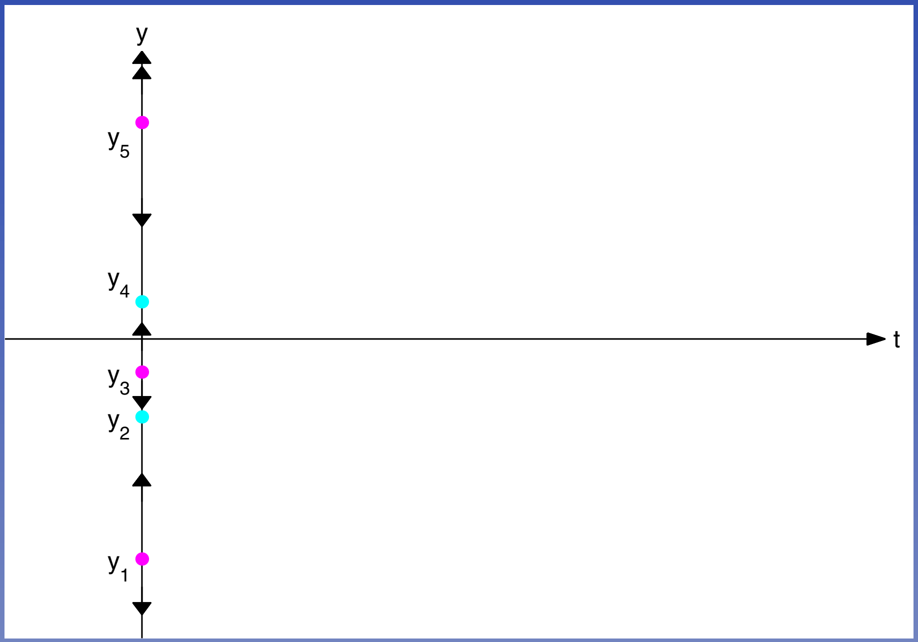
Qualitative Analysis of $y' = f(y)$.

4. Analyze the equilibrium points.



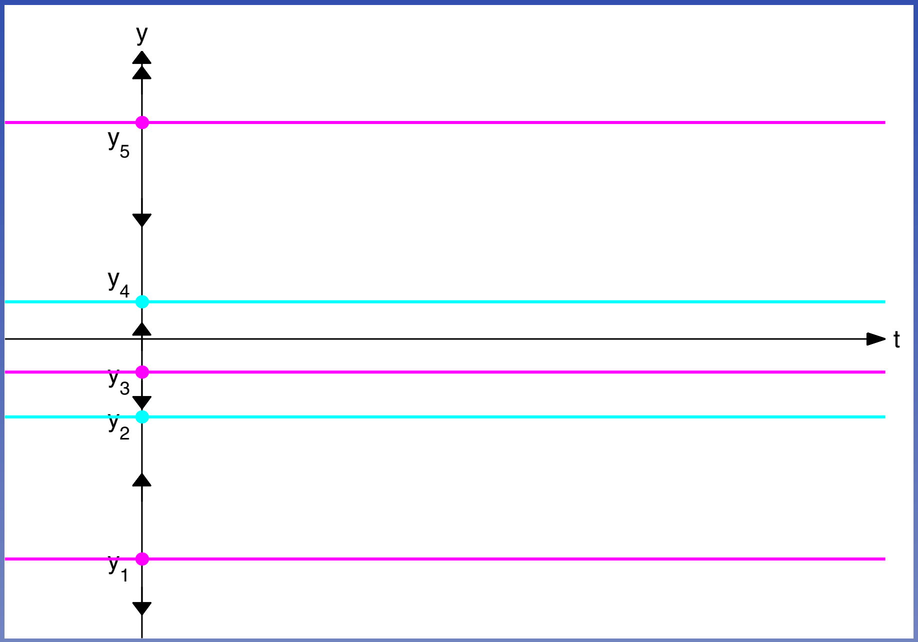
Qualitative Analysis of $y' = f(y)$.

5. Transfer the phase line to ty-space.



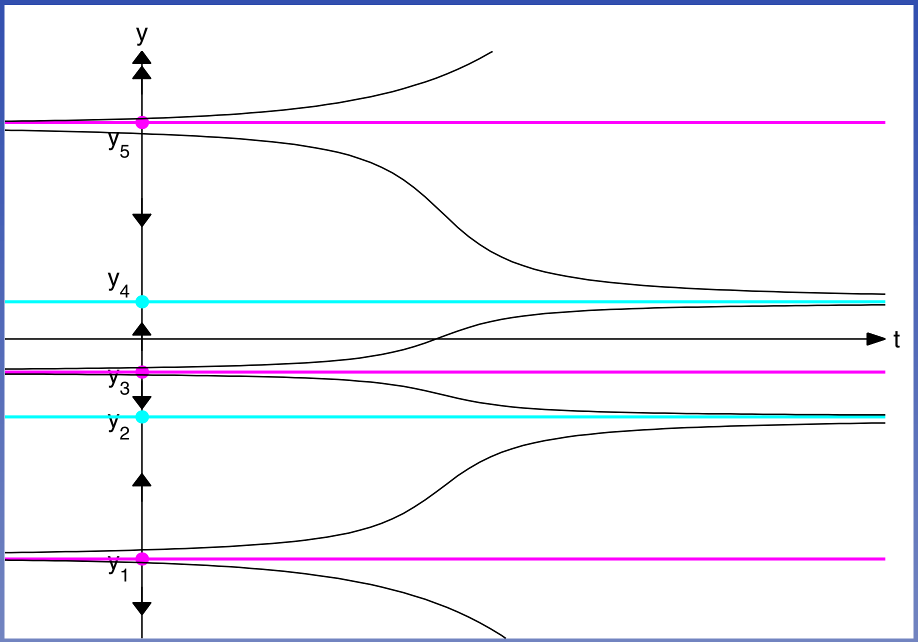
Qualitative Analysis of $y' = f(y)$.

6. Plot the equilibrium solutions.



Qualitative Analysis of $y' = f(y)$.

7. Plot other solutions approximately.



Seven Steps

1. Graph $y \longrightarrow f(y)$.
2. Find the equilibrium points where $f(y) = 0$.
3. Determine the behavior between eq. pts.
4. Analyze the equilibrium points.
5. Transfer the phase line to ty-space.
6. Plot the equilibrium solutions.
7. Plot other solutions approximately.