Math 211

Lecture #15

Systems of Linear Equations

October 1, 2001

Example

Solve

$$3x - 4y + 5z = 3$$
$$-x + 2y - 2z = -2$$

- Find *all* solutions.
- Find a systematic method which works for all systems, no matter how large.

Vectors and Matrices

- Introduce the vectors

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

and the matrix

$$C = \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix}$$

- $\mathbf{x}$ is the *vector of unknowns*, $\mathbf{b}$ is the *RHS*, and C is the *coefficient matrix*,
- We will define the product $C\mathbf{x}$ so that the system can be written as $C\mathbf{x} = \mathbf{b}$.

Vectors

- A vector is a list of numbers
- 2-vectors, 3-vectors, n-vectors
- Row vectors and column vectors.
- A vector has length and direction
  - Parallel vectors are equal
- Transpose of a vector, $\mathbf{v}^T$.

Algebra of Vectors

- Addition of Vectors
  - Algebraic view of addition
  - Geometric view of addition
  - Addition of more than two vectors
- Multiplication by a Scalar
  - Algebraic view
  - Geometric view

Linear Combinations of Vectors

- Vectors $\mathbf{x} = (2, -3)^T$ and $\mathbf{y} = (-1, 2)^T$.
- Any vector of the form $a\mathbf{x} + b\mathbf{y}$ is a *linear combination* of $\mathbf{x}$ and $\mathbf{y}$.
- $3\mathbf{x} + 2\mathbf{y} = (4, -5)^T$.
- Any 2-vector is a linear combination of $\mathbf{x}$ and $\mathbf{y}$.
- Linear combinations of more than two vectors.

Matrices

- A matrix is a rectangular array of numbers.
- Example

$$A = \begin{pmatrix} -1 & 0 & 2 & 6 \\ 0 & 3 & -4 & 10 \\ 3 & 3 & 2 & -5 \end{pmatrix}$$

- Size of $A =$ (3,4); 3 rows & 4 columns.
  - 3 row vectors and 4 column vectors.

Solution method

Linear Combinations and Systems

- The example system can be written as a vector equation

$$\begin{pmatrix} 3x - 4y + 5z \\ -x + 2y - 2z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

- or

$$x \begin{pmatrix} 3 \\ -1 \end{pmatrix} + y \begin{pmatrix} -4 \\ 2 \end{pmatrix} + z \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

- These vectors are the column vectors in the coefficient matrix.

Return

Coefficient Matrix

- The coefficient matrix is

$$C = \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix}$$

- Solving the system of equations $\Leftrightarrow$ finding a linear combination of the columns of the coefficient matrix which is equal to the RHS.

Return Linear combination

Product of a Matrix with a Vector

- The *product* of a matrix A and a vector $\mathbf{x}$ is the linear combination of the columns of A with the elements of $\mathbf{x}$ as coefficients.
- Example:

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= x \begin{pmatrix} 3 \\ -1 \end{pmatrix} + y \begin{pmatrix} -4 \\ 2 \end{pmatrix} + z \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

Return Coefficient matrix Linear Comb. System

Example

- Thus the system of equations becomes

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

or

$$C\mathbf{x} = \mathbf{b}$$

Product Coefficient matrix Linear Comb. Solution

Computing the Product of a Matrix and a Vector.

- From the definition.
- A faster way.
  - $A = (a_{ij})$, a $p \times q$ matrix, and $\mathbf{x}$, a column q-vector.

$$A\mathbf{x} = \mathbf{y} \Leftrightarrow$$
$$y_i = \sum_{j=1}^{q} a_{ij} x_j \quad \text{for } 1 \le i \le p.$$

- $A\mathbf{x}$ is only defined if A has the same number of columns as $\mathbf{x}$ has rows.

Return

Algebraic Properties of the Matrix-Vector Product

Suppose A is a matrix, $\mathbf{x}$ and $\mathbf{y}$ are vectors, and a and b are numbers.

- $A(a\mathbf{x}) = a(A\mathbf{x})$
- $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$
- $A(a\mathbf{x} + b\mathbf{y}) = aA\mathbf{x} + bA\mathbf{y}$
- Multiplication by a matrix is a *linear operation.*

Product of Two Matrices

Suppose A is $n \times p$ and B is $p \times q$.
Write B in terms of its column vectors

$$B = [\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_q]$$

Define the *product* AB by

$$AB = [A\mathbf{b}_1\ A\mathbf{b}_2\ \ldots\ A\mathbf{b}_q]$$

Algebraic Properties of the Product

Suppose that A, B, and C are matrices

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- However $AB \neq BA$ in general

The Identity Matrix

- In dimension 3
$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
- $I\mathbf{x} = \mathbf{x}$ for every 3-vector $\mathbf{x}$.
- $IA = A$ for every matrix A with 3 rows.
- $AI = A$ for every matrix A with 3 columns.