# Math 211

Lecture #16

Geometry of Solution Sets

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# Systems and Solutions

- The system

$$\begin{aligned} 3x + 4y - 5z &= 3 \\ -2x + 3z &= -7 \end{aligned} \Leftrightarrow A\mathbf{x} = \mathbf{b}$$

  where

$$A = \begin{pmatrix} 3 & 4 & -5 \\ -2 & 0 & 3 \end{pmatrix},$$

$$\mathbf{x} = (x, y, z)^T, \quad \mathbf{b} = (3, -7)^T.$$

- A *solution* is a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. $(2, -2, -1)^T$ is a solution.

# Solution Sets

- The *solution set* is set of *all* solutions to a system of linear equations.
  - What kinds of sets can be solution sets?
  - Can a circle be the solution set for a system of linear equations?
- We will examine *all* possibilities in 2 and 3 dimensions.
- Geometry will tell us the answer.

# One Equation in Two Variables

Example: $2x - 3y = 1$

- The solution set is a line in the plane.
- Solve for $y$ : $y = (-1 + 2x)/3$
- The solution set consists of all vectors $(x, y)^T$, where $y = (-1 + 2x)/3$.

# The Solution Set

- consists of all vectors of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ (-1+2x)/3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -1/3 \end{pmatrix} + \begin{pmatrix} x \\ 2x/3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -1/3 \end{pmatrix} + x \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$$

- $x$ is a free parameter.

# Parametric Equation for a Line

$$\mathbf{u} = \mathbf{u}_0 + x\mathbf{v}$$

- In our case $\mathbf{u}_0 = (0,\ -1/3)^T$ and $\mathbf{v} = (1,\ 2/3)^T$
- The vector $\mathbf{u}_0$ locates one point on the line.
- The vector $\mathbf{v}$ gives the direction of the line.
- The number $x$ tells how far the point $\mathbf{u}$ is from $\mathbf{u}_0$.

# Two Equations in Two Variables

Example:

$$2x - 3y = 1$$

$$x + y = 3$$

- In matrix form

$$\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

# Two Equations — Two Lines

- In general there are three possibilities
- In this case the lines intersect in one point $(2,\ 1)^T$.
- Other possibilities:
  - The two lines are the same line, and intersect in a line.
  - The two lines are parallel, and the intersection is empty. Such equations are *inconsistent*.

# Possible Solution Sets in Dimension 2

Three possibilities:

- The empty set.
- A single point.
- A line.
- Can a circle be the solution set for a system of linear equations?

# One Equation in Three Variables

Example: $2x - 3y + 4z = 1$

- Solution set is a plane in 3-space.
- Solve for $z$ : $z = (1 - 2x + 3y)/4$.
- The solution set is all vectors $(x, y, z)^T$, where $z = (1 - 2x + 3y)/4$.

# The Solution Set

- consists of all vectors of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ (1-2x+3y)/4 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} + \begin{pmatrix} x \\ y \\ -x/2+3y/4 \end{pmatrix}$$

# The Solution Set (cont)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} + x \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 3/4 \end{pmatrix}$$

- $x$ and $y$ are free parameters.

# Parametric Equation for a Plane

$$\mathbf{u} = \mathbf{u}_0 + x\mathbf{v} + y\mathbf{w}$$

- In our case $\mathbf{u}_0 = (0,\ 0,\ 1/4)^T$, $\mathbf{v} = (1,\ 0,\ -1/2)^T$, and $\mathbf{w} = (0,\ 1,\ 3/4)^T$
- $\mathbf{u}_0$ locates one point on the plane.
- $\mathbf{v}$ and $\mathbf{w}$ give two different directions in the plane.
- $\mathbf{u}$ differs from $\mathbf{u}_0$ by the linear combination of $\mathbf{v}$ and $\mathbf{w}$ with coefficients $x$ & $y$.

# Two Equations in Three Variables

Example:

$$2x - 3y + 4z = 1$$

$$x + y - z = 3$$

- In matrix form

$$\begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Return

# Two Equations — Two Planes

- In general there are three possibilities — $\emptyset$, a line, or a plane.
- In this case the two planes intersect in a line.
- Solve for $z$ & $y$ in terms of $x$:
  $y = 13 - 6x$ and $z = 10 - 5x$

Return

# The Solution Set

- consissts of all vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 13 - 6x \\ 10 - 5x \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 13 \\ 10 \end{pmatrix} + x \begin{pmatrix} 1 \\ -6 \\ -5 \end{pmatrix}$$

- This is a line in 3-space.

# Three Equations in Three Variables

Example:

$$\begin{aligned} 2x - 3y + 4z &= 1 \\ x + y - z &= 3 \\ 3x - y + 3z &= 5 \end{aligned}$$

- In matrix form

$$\begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \\ 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

# Three Equations — Three Planes

- In general there are four possibilities — $\emptyset$, a point, a line, or a plane.
- In this case the three planes intersect in a point.
- Solve to find that $(x, y, z)^T = (2,\ 1,\ 0)^T$

# Possible Solution Sets in Dimension 3

Four possibilities:

- The empty set.
- A single point.
- A line.
- A plane.
- Can a sphere be a solution set of a system of linear equations?

# Solution Sets in Higher Dimension

By analogy with dimensions 2 &3, we expect

- The solution set could be $\emptyset$ or a point.
- If a solution set contains 2 points, then it contains the line through them.
- If a solution set contains 3 points not on the same line, then it contains the plane though them.

# Solution Sets of Homogeneous Equations

- $\mathbf{0}$ is the vector with all entries $= 0$. $\mathbf{0}$ is referred to as the *0 vector* or the *origin.*
- A *homogeneous system* is one of the form

$$A\mathbf{x} = \mathbf{0}.$$

Example:

$$\begin{aligned} 2x - 3y + 4z &= 0 \\ x + y - z &= 0 \\ 3x - y + 3z &= 0 \end{aligned}$$

- A homogeneous system always has $\mathbf{0}$ as a solution.
- Hence the solution set of a homogeneous system is never the empty set.