Math 211

Lecture #17

Solving Systems of Equations

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Solving Systems of Equations

- We want to find a way to find the solution set of any system.
- We will build towards the method by looking at a series of examples.

Example

$$x + y = 3$$

$$2x - 3y = 1$$

- Solve the first equation for x and substitute into second equation. We get the system

$$x + y = 3$$

$$-5y = -5$$

Comparison

- The two systems have the same solutions.
- The second is very easy to solve because the variable x has been *eliminated.*
- Solve last equation first, then the first equation. This is called *backsolving*.
  - $y = 1$, then $x = 2$.
- Elimination and backsolving is our method.

Example (reprise)

$$x + y = 3$$
$$2x - 3y = 1$$

- Add -2 times the first equation to the second equation to eliminate x. We get the system

$$x + y = 3$$
$$-5y = -5$$

- Solve by backsolving: $y = 1$, then $x = 2$.

Example — Using Matrix Notation

$$\begin{aligned} x + y &= 3 \\ 2x - 3y &= 1 \end{aligned} \qquad \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

- Form the *augmented matrix*

$$M = \begin{pmatrix} 1 & 1 & 3 \\ 2 & -3 & 1 \end{pmatrix}$$

- Each row in M contains all of the information about one of the equations in the system.

Example (continued)

$$\begin{aligned} x + y &= 3 \\ 2x - 3y &= 1 \end{aligned} \Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 2 & -3 & 1 \end{pmatrix}$$

- Add -2 times the first row to the second row, eliminating the coefficient of x.

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & -5 & -5 \end{pmatrix} \Rightarrow \begin{aligned} x + y &= 3 \\ -5y &= -5 \end{aligned}$$

- Backsolve.

Method of Solution

- Write down the augmented matrix.
- Eliminate as many coefficients as possible.
  - This is not well defined yet.
- Write down the simplified system.
- Solve the simplified system by backsolving.

Example

$$x + y - z = 3$$
$$2x - 3y + 4z = 1$$

- Augmented matrix:

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -3 & 4 & 1 \end{pmatrix}$$

- Add -2 times the first row to the second row to eliminate the coefficient of x.

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -5 & 6 & -5 \end{pmatrix}$$

- Simplified system:

$$x + y - z = 3$$
$$-5y + 6z = -5$$

- Backsolve

$$x + y - z = 3$$

$$-5y + 6z = -5$$

- ♦ z is a free variable. Set $z = t$.
- ♦ Solve for $y = 1 + 6t/5$.
- ♦ Solve for $x = 2 - t/5$.

- Solutions are the vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - t/5 \\ 1 + 6t/5 \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/5 \\ 6/5 \\ 1 \end{pmatrix}$$

- The solution set is a line in $\mathbf{R}^3$.

Elimination — Equations

We only use operations on the equations which will lead to systems of equations with the same solutions.

- Add a multiple of one equation to another.
- Interchange two equations.
- Multiply an equation by a non-zero number.

Elimination — Row operations

The corresponding operations on the rows of the augmented matrix are called *row operations*.

- Add a multiple of one row to another.
- Interchange two rows.
- Multiply a row by a non-zero number.

The Goal of Elimination

- How simple can we make the augmented matrix?

$$\begin{pmatrix} P & * & * & * & * & * & * & * \\ 0 & P & * & * & * & * & * & * \\ 0 & 0 & 0 & P & * & * & * & * \\ 0 & 0 & 0 & 0 & P & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & P & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- P is a nonzero number, $*$ is any number.

Row Echelon Form

- The *pivot* of a row is the first non-zero element from the left.
- A matrix is in *row echelon form* if every pivot lies strictly to the right of those in rows above.

Reduced Row Echelon Form

- Row echelon form, plus all pivots $= 1$ and all other entries in a pivot column are 0.

$$\begin{pmatrix} 1 & 0 & * & 0 & 0 & * & 0 & 0 \\ 0 & 1 & * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

$$3x_2 - 4x_3 = -7$$

$$-x_1 + 2x_2 = -3$$

$$3x_1 + 2x_2 + x_3 = 2$$

- Augmented matrix:

$$\begin{pmatrix} 0 & 3 & -4 & -7 \\ -1 & 2 & 0 & -3 \\ 3 & 2 & 1 & 2 \end{pmatrix}$$

- Elimination: $\begin{pmatrix} -1 & 2 & 0 & -3 \\ 0 & 3 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
- Simplified system:

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 3x_2 - 4x_3 &= -7 \\ x_3 &= 1 \end{aligned}$$

- Backsolve: $x_3 = 1$, $x_2 = -1$, and $x_1 = 1$.

Example

$$A = \begin{pmatrix} 1 & 2 & 5 & -1 \\ 1 & 2 & -3 & 8 \\ 3 & 6 & 7 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ -12 \\ -16 \end{pmatrix}$$

- System $A\mathbf{x} = \mathbf{b}$
- Augmented matrix:

$$M = [A,\ \mathbf{b}] = \begin{pmatrix} 1 & 2 & 5 & -1 & -2 \\ 1 & 2 & -3 & 8 & -12 \\ 3 & 6 & 7 & 6 & -16 \end{pmatrix}$$

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- Elimination using MATLAB:

$$\begin{pmatrix} 1 & 2 & 5 & -1 & -2 \\ 0 & 0 & -8 & 9 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Simplified system:

$$\begin{aligned} x_1 + 2x_2 + 5x_3 - x_4 &= -2 \\ -8x_3 + 9x_4 &= -10 \end{aligned}$$

- Backsolving:
  - There are pivots in columns 1 & 3. These are *pivot columns*. The corresponding variables x_1 and x_3 are called *pivot variables*.
  - The other columns are called *free columns*.The corresponding variables x_2 and x_4 are called *free variables*.

- The free variables may be assigned arbitrary values: $x_2 = s$ and $x_4 = t$.
- Backsolve for the pivot variables.

$$
\begin{aligned}
x_3 &= (10 + 9x_4)/8 = 5/4 + 9t/8 \\
x_1 &= -2 - 2x_2 - 5x_3 + x_4 \\
&= -2 - 2s - 5(5/4 + 9t/8) + t \\
&= -33/4 - 2s - 37t/8
\end{aligned}
$$

Return

- The solutions are the vectors

$$\mathbf{x} = \begin{pmatrix} -33/4 \\ 0 \\ 5/4 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -37/8 \\ 0 \\ 9/8 \\ 0 \end{pmatrix}$$

- The solution set is a plane in $\mathbf{R}^4$.

Method of Solution for $A\mathbf{x} = \mathbf{b}$

- Use the augmented matrix $M = [A,\ \mathbf{b}]$.
- Eliminate as many coefficients as possible.
  - Use row operations to get to row echelon form.
- Write down the simplified system.
- Backsolve.
  - Assign arbitrary values to the free variables.
  - Backsolve for the pivot variables.