Math 211

Lecture #18

Properties of Solution Sets

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Method of Solution for $A\mathbf{x} = \mathbf{b}$

- Use the augmented matrix $M = [A,\ \mathbf{b}]$.
- Eliminate as many coefficients as possible.
  - Use row operations to reduce to row echelon form.
- Write down the simplified system.
- Backsolve.
  - Assign arbitrary values to the free variables.
  - Solve for the pivot variables.

Consistent Systems

- A system is *consistent* if it has solutions.
    - The solution set is not the empty set.
- A system is consistent if and only if the simplified system is consistent.
- This is true if and only if the last column (after elimination) does *not* contain a pivot.

Examples

$$A = \begin{pmatrix} -3 & 6 & 0 \\ -2 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix} \quad \mathbf{b}_1 = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \quad \mathbf{b}_2 = \begin{pmatrix} -9 \\ -6 \\ 7 \end{pmatrix}$$

- Use `A1`, `bb1`, & `bb2`

Homogeneous Systems

Example $A\mathbf{x} = \mathbf{0}$.

$$A = \begin{pmatrix} -5 & -4 & -2 \\ -5 & -6 & -2 \\ 30 & 27 & 11 \end{pmatrix} \Rightarrow \begin{pmatrix} -5 & -4 & -2 & 0 \\ -5 & -6 & -2 & 0 \\ 30 & 27 & 11 & 0 \end{pmatrix}$$

- Use `A2`
- During elimination the column of zeros is unchanged.
- It is unnecessary to augment a homogeneous system.

Square Matrices

- There are special kinds:
  - Singular and nonsingular.
  - Invertible and noninvertible.
- What do the terms mean?
- What are the relations between them?

Singular and Nonsingular Matrices

The $n \times n$ matrix A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side $\mathbf{b}$.

Proposition: The $n \times n$ matrix A is nonsingular if and only if the simplified matrix has only nonzero entries along the diagonal.

- In reduced row echelon form we get I.

Examples

$$A = \begin{pmatrix} -17 & -16 & -6 \\ 18 & 18 & 6 \\ 6 & 3 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} -17 & -16 & -6 \\ 18 & 18 & 6 \\ 6 & 3 & 4 \end{pmatrix}$$

- Use `A3`

Proposition: If the $n \times n$ matrix A is nonsingular then the equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for any right hand side $\mathbf{b}$.

Proposition: The $n \times n$ matrix A is singular if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.

- This is a result that we will use repeatedly.

Invertible Matrices

An $n \times n$ matrix A is *invertible* if there is an $n \times n$ matrix B such that $AB = BA = I$. The matrix B is called an *inverse* of A.

- If B_1 and B_2 are both inverses of A, then

$$B_1 = B_1(AB_2) = (B_1A)B_2 = B_2$$

- The inverse of A is denoted by A^{-1}.
- Invertible $\Rightarrow$ nonsingular .

Computing the inverse A^{-1}

- Form the matrix $[A,\ I]$.
- Do elimination until the matrix has the form $[I,\ B]$.
- Then $A^{-1} = B$.
- A matrix is invertible if and only if it is nonsingular.
- Example $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- Use `A3`

Structure of the Solution Set

Theorem: Let $\mathbf{x}_p$ be a particular solution to $A\mathbf{x}_p = \mathbf{b}$.

1. If $A\mathbf{x}_h = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ also satisfies $A\mathbf{x} = \mathbf{b}$.
2. If $A\mathbf{x} = \mathbf{b}$, then there is a vector$\mathbf{x}_h$ such that $A\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

- Solution set for $A\mathbf{x} = \mathbf{b}$ is known if we know one particular solution $\mathbf{x}_p$ and the solution set for the homogeneous system $A\mathbf{x}_h = \mathbf{0}$.

Solution Set of a Homogeneous System

Our goal is to understand such sets better. In particular we want to know:

- What are the properties of these solution sets?
- Is there a convenient way to describe them?

Nullspace of a Matrix

The *nullspace* of a matrix A is the set

$$\{\mathbf{x} \,|\, A\mathbf{x} = \mathbf{0}\}\,.$$

- The nullspace of A is the same as the solution set for the homogeneous system $A\mathbf{x} = \mathbf{0}$.
- The nullspace of A is denoted by $\mathrm{null}(A)$,

Properties of the Nullspace of A

Proposition: Let A be a matrix.

1. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathrm{null}(A)$, then $\mathbf{x} + \mathbf{y}$ is in $\mathrm{null}(A)$.
2. If a is a scalar and $\mathbf{x}$ is in $\mathrm{null}(A)$, then $a\mathbf{x}$ is in $\mathrm{null}(A)$.

- $\mathrm{null}(A)$ has some of the same properties as $\mathbf{R}^n$.

Subspaces of $\mathbf{R}^n$

Definition: A nonempty subset V of $\mathbf{R}^n$ that has the properties

1. if $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, $\mathbf{x}+\mathbf{y}$ is in V,
2. if a is a scalar, and $\mathbf{x}$ is in V, then $a\mathbf{x}$ is in V,

is called a *subspace* of $\mathbf{R}^n$.

- The nullspace of a matrix is a subspace.

Return

Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through the origin is a subspace.
  $V = \{t\mathbf{v} \,|\, t \in \mathbf{R}\}$.
- A plane through the origin is a subspace.
  $V = \{a\mathbf{v} + b\mathbf{w} \,|\, a,\, b \in \mathbf{R}\}$.
- $\{\mathbf{0}\}$ and $\mathbf{R}^n$ are subspaces of $\mathbf{R}^n$.
  - These are the *trivial subspaces.*

Linear Combinations

Proposition: Any linear combination of vectors in a subspace V is also in V.

- Subspaces of $\mathbf{R}^n$ have the same kind of linear structure as $\mathbf{R}^n$ itself.
- In particular the nullspaces of matrices have the same kind of linear structure as $\mathbf{R}^n$.

Row operations

The permissable operations on the rows of the augmented matrix are called *row operations*.

- Add a multiple of one row to another.
- Interchange two rows.
- Multiply a row by a non-zero number.

Row Echelon Form

A matrix is in *row echelon form* if every pivot lies strictly to the right of those in rows above.

$$\begin{pmatrix} P & * & * & * & * & * & * & * & * \\ 0 & P & * & * & * & * & * & * & * \\ 0 & 0 & 0 & P & * & * & * & * & * \\ 0 & 0 & 0 & 0 & P & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & P & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- P is a pivot, $*$ is any number.