

# Math 211

Lecture #19

Nullspaces and Subspaces

October 10, 2001

## Structure of the Solution Set

**Theorem:** Let  $\mathbf{x}_p$  be a particular solution to  $A\mathbf{x}_p = \mathbf{b}$ .

1. If  $A\mathbf{x}_h = \mathbf{0}$  then  $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$  also satisfies  $A\mathbf{x} = \mathbf{b}$ .
  2. If  $A\mathbf{x} = \mathbf{b}$ , then there is a vector  $\mathbf{x}_h$  such that  $A\mathbf{x}_h = \mathbf{0}$  and  $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ .
- The solution set for  $A\mathbf{x} = \mathbf{b}$  is known if we know one particular solution  $\mathbf{x}_p$  and the solution set for the homogeneous system  $A\mathbf{x}_h = \mathbf{0}$ .

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## Solution Set of a Homogeneous System

Our goal is to understand such sets better. In particular we want answers to the following questions.

- What are the properties of these solution sets?
- Is there a convenient way to describe them?

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[Solution set](#)

## Nullspace of a Matrix

The *nullspace* of a matrix  $A$  is the set

$$\text{null}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}.$$

- The nullspace of  $A$  is the same as the solution set for the homogeneous system  $A\mathbf{x} = \mathbf{0}$ .

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## Properties of a Nullspace

**Proposition:** Let  $A$  be a matrix.

1. If  $\mathbf{x}$  and  $\mathbf{y}$  are in  $\text{null}(A)$ , then  $\mathbf{x} + \mathbf{y}$  is in  $\text{null}(A)$ .
2. If  $a$  is a scalar and  $\mathbf{x}$  is in  $\text{null}(A)$ , then  $a\mathbf{x}$  is in  $\text{null}(A)$ .

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## Subspaces of $\mathbf{R}^n$

**Definition:** A nonempty subset  $V$  of  $\mathbf{R}^n$  that has the properties

1. if  $\mathbf{x}$  and  $\mathbf{y}$  are vectors in  $V$ ,  $\mathbf{x} + \mathbf{y}$  is in  $V$ ,
2. if  $a$  is a scalar, and  $\mathbf{x}$  is in  $V$ , then  $a\mathbf{x}$  is in  $V$ ,

is called a *subspace* of  $\mathbf{R}^n$ .

- A nullspace of a matrix is a subspace.

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## Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through  $\mathbf{0}$  is a subspace.  $V = \{t\mathbf{v} \mid t \in \mathbf{R}\}$ .
- A plane through  $\mathbf{0}$  is a subspace.  
 $V = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\}$ .
- $\{\mathbf{0}\}$  and  $\mathbf{R}^n$  are subspaces of  $\mathbf{R}^n$ .
  - ♦ These are called the *trivial subspaces*.

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## Linear Combinations

**Proposition:** Any linear combination of vectors in a subspace  $V$  is also in  $V$ .

- Subspaces of  $\mathbf{R}^n$  have the same kind of linear structure as  $\mathbf{R}^n$  itself.
- In particular nullspaces of matrices have the same kind of linear structure as  $\mathbf{R}^n$ .

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## Example 1

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

The nullspace of  $A$  is

$$\text{null}(A) = \{a\mathbf{v} \mid a \in \mathbf{R}\},$$

where  $\mathbf{v} = (1, -1, 1)^T$ .

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## Example 2

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

- $\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\}$ , where  $\mathbf{v} = (1, -1, 1, 0)^T$  and  $\mathbf{w} = (0, -2, 0, 1)^T$ .
- $\text{null}(B)$  consists of all linear combinations of  $\mathbf{v}$  and  $\mathbf{w}$ .

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## The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

**Definition:** The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).$$

**Proposition:** If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are all vectors in  $\mathbf{R}^n$ , then  $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$  is a subspace of  $\mathbf{R}^n$ .

Return

null(A)

null(B)

Examples

Nullspace

## When is $\mathbf{w}$ in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ ?

$\Leftrightarrow$   $\mathbf{w}$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ ?

$\Leftrightarrow$  There are constants  $a_1, a_2, \dots, a_k$  such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = \mathbf{w}.$$

$\Leftrightarrow$  There is a solution  $\mathbf{a} = (a_1, a_2, \dots, a_k)^T$  to the system

$$V\mathbf{a} = \mathbf{w},$$

where  $V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ .

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Span

## Examples

Let  $\mathbf{v}_1 = (1, 2)^T$ ,  $\mathbf{v}_2 = (1, 0)^T$ , and  $\mathbf{v}_3 = (2, 0)^T$ .

- $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{R}^2$ . (Proof?)
- $\text{span}(\mathbf{v}_1, \mathbf{v}_3) = \mathbf{R}^2$ . (Proof?)
- $\text{span}(\mathbf{v}_2, \mathbf{v}_3) = \text{span}(\mathbf{v}_2)$ . (Proof?)
  - ♦  $\text{span}(\mathbf{v}_2, \mathbf{v}_3) = \{t\mathbf{v}_2 \mid t \in \mathbf{R}\}$ .
  - ♦  $\mathbf{v}_2$  and  $\mathbf{v}_3$  have the same direction.

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Span

## Linear Independence of Two Vectors

We need a condition that will keep unneeded vectors out of a spanning list. We will work toward a general definition.

- Two vectors are *linearly dependent* if one is a scalar multiple of the other.
  - ♦  $\mathbf{v}_2$  and  $\mathbf{v}_3$  are linearly dependent.
  - ♦  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are *linearly independent*.

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## Linear Independence of Three Vectors

- Three vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  are *linearly dependent* if one is a linear combination of the other two.
  - ♦ Example:  $\mathbf{v}_1 = (1, 0, 0)^T$ ,  $\mathbf{v}_2 = (0, 1, 0)^T$ , and  $\mathbf{v}_3 = (1, 2, 0)^T$

$$\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2.$$

- ♦ Notice that  $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$ .

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## Linear Independence

- Three vectors are linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
  - ♦ Non-trivial means that at least one of the coefficients is not 0.
- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.

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## Linear Independence

**Definition:** The vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots,$  and  $\mathbf{v}_k$  are *linearly independent* if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k = \mathbf{0}$$

$$\Rightarrow c_1 = c_2 = \dots = c_k = 0.$$

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[Two vectors](#)

[Three vectors](#)

[More vectors](#)

## Basis of a Subspace

**Definition:** A set of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots,$  and  $\mathbf{v}_k$  form a *basis* of a subspace  $V$  if

1.  $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$
2.  $\mathbf{v}_1, \mathbf{v}_2, \dots,$  and  $\mathbf{v}_k$  are linearly independent.

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[Span](#)

## Examples of Bases

- The vector  $\mathbf{v} = (1, -1, 1)^T$  is a basis for  $\text{null}(A)$ .
  - ♦  $\text{null}(A)$  is the subspace of  $\mathbf{R}^3$  with basis  $\mathbf{v}$ .
- The vectors  $\mathbf{v} = (1, -1, 1, 0)^T$  and  $\mathbf{w} = (0, -2, 0, 1)^T$  form a basis for  $\text{null}(B)$ .
  - ♦  $\text{null}(B)$  is the subspace of  $\mathbf{R}^4$  with basis  $\{\mathbf{v}, \mathbf{w}\}$ .

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Examples

Nullspace

## Basis of a Subspace

**Proposition:** Let  $V$  be a subspace of  $\mathbf{R}^n$ .

1. If  $V \neq \{0\}$ , then  $V$  has a basis.
2. Every basis of  $V$  has the same number of elements.

**Definition:** The *dimension* of a subspace  $V$  is the number of elements in a basis of  $V$ .

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Examples

Examples

Nullspace

## Linear Independence?

How do we decide if a set of vectors is linearly independent?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}$$

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$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Leftrightarrow$$

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]\mathbf{c} = \mathbf{0} \Leftrightarrow$$

$$\mathbf{c} \in \text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]).$$

- $\mathbf{c} = (-3, 2, 1)^T \in \text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3])$ ,  $\Rightarrow$   
 $-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$
- $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly dependent.

Return

Example

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 3 \end{pmatrix}$$

- $\text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = \{\mathbf{0}\}.$
- $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent.

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Method

**Proposition:** Suppose that  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are vectors in  $\mathbf{R}^n$ . Set  $V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ .

1. If  $\text{null}(V) = \{\mathbf{0}\}$ , then  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly independent.
2. If  $\mathbf{c} = (c_1, c_2, \dots, c_k)^T$  is a nonzero vector in  $\text{null}(V)$ , then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0},$$

so the vectors are linearly dependent.