

Math 211

Lecture #19

Nullspaces and Subspaces

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Structure of the Solution Set

Theorem: Let \mathbf{x}_p be a particular solution to $A\mathbf{x}_p = \mathbf{b}$.

1. If $A\mathbf{x}_h = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ also satisfies $A\mathbf{x} = \mathbf{b}$.
 2. If $A\mathbf{x} = \mathbf{b}$, then there is a vector \mathbf{x}_h such that $A\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.
- The solution set for $A\mathbf{x} = \mathbf{b}$ is known if we know one particular solution \mathbf{x}_p and the solution set for the homogeneous system $A\mathbf{x}_h = \mathbf{0}$.

Solution Set of a Homogeneous System

Our goal is to understand such sets better. In particular we want answers to the following questions.

- What are the properties of these solution sets?
- Is there a convenient way to describe them?

Nullspace of a Matrix

The *nullspace* of a matrix A is the set

$$\text{null}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}.$$

- The nullspace of A is the same as the **solution set** for the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Properties of a Nullspace

Proposition: Let A be a matrix.

1. If \mathbf{x} and \mathbf{y} are in $\text{null}(A)$, then $\mathbf{x} + \mathbf{y}$ is in $\text{null}(A)$.
2. If a is a scalar and \mathbf{x} is in $\text{null}(A)$, then $a\mathbf{x}$ is in $\text{null}(A)$.

Subspaces of \mathbf{R}^n

Definition: A nonempty subset V of \mathbf{R}^n that has the properties

1. if \mathbf{x} and \mathbf{y} are vectors in V , $\mathbf{x} + \mathbf{y}$ is in V ,
2. if a is a scalar, and \mathbf{x} is in V , then $a\mathbf{x}$ is in V ,

is called a *subspace* of \mathbf{R}^n .

- A *nullspace* of a matrix is a subspace.

Examples of Subspaces

- The **nullspace** of a matrix is a **subspace**.
- A line through $\mathbf{0}$ is a subspace. $V = \{t\mathbf{v} \mid t \in \mathbf{R}\}$.
- A plane through $\mathbf{0}$ is a subspace.
 $V = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\}$.
- $\{\mathbf{0}\}$ and \mathbf{R}^n are subspaces of \mathbf{R}^n .
 - ♦ These are called the *trivial subspaces*.

Linear Combinations

Proposition: Any linear combination of vectors in a subspace V is also in V .

- Subspaces of \mathbf{R}^n have the same kind of linear structure as \mathbf{R}^n itself.
- In particular **nullspaces** of matrices have the same kind of linear structure as \mathbf{R}^n .

Example 1

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

The nullspace of A is

$$\text{null}(A) = \{a\mathbf{v} \mid a \in \mathbf{R}\},$$

where $\mathbf{v} = (1, -1, 1)^T$.

Example 2

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

- $\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\}$, where $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$.
- $\text{null}(B)$ consists of all linear combinations of \mathbf{v} and \mathbf{w} .

The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

Definition: The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).$$

Proposition: If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are all vectors in \mathbf{R}^n , then $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ is a subspace of \mathbf{R}^n .

When is \mathbf{w} in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$?

\Leftrightarrow \mathbf{w} is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k ?

\Leftrightarrow There are constants a_1, a_2, \dots, a_k such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = \mathbf{w}.$$

\Leftrightarrow There is a solution $\mathbf{a} = (a_1, a_2, \dots, a_k)^T$ to the system

$$V\mathbf{a} = \mathbf{w},$$

where $V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$.

Examples

Let $\mathbf{v}_1 = (1, 2)^T$, $\mathbf{v}_2 = (1, 0)^T$, and $\mathbf{v}_3 = (2, 0)^T$.

- $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{R}^2$. (Proof?)
- $\text{span}(\mathbf{v}_1, \mathbf{v}_3) = \mathbf{R}^2$. (Proof?)
- $\text{span}(\mathbf{v}_2, \mathbf{v}_3) = \text{span}(\mathbf{v}_2)$. (Proof?)
 - ♦ $\text{span}(\mathbf{v}_2, \mathbf{v}_3) = \{t\mathbf{v}_2 \mid t \in \mathbf{R}\}$.
 - ♦ \mathbf{v}_2 and \mathbf{v}_3 have the same direction.

Linear Independence of Two Vectors

We need a condition that will keep unneeded vectors out of a spanning list. We will work toward a general definition.

- Two vectors are *linearly dependent* if one is a scalar multiple of the other.
 - ◆ \mathbf{v}_2 and \mathbf{v}_3 are linearly dependent.
 - ◆ \mathbf{v}_1 and \mathbf{v}_2 are *linearly independent*.

Linear Independence of Three Vectors

- Three vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 are *linearly dependent* if one is a linear combination of the other two.
 - ♦ Example: $\mathbf{v}_1 = (1, 0, 0)^T$, $\mathbf{v}_2 = (0, 1, 0)^T$, and $\mathbf{v}_3 = (1, 2, 0)^T$

$$\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2.$$

- ♦ Notice that $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

Linear Independence

- **Three vectors** are linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
 - ◆ Non-trivial means that at least one of the coefficients is not 0.
- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.

Linear Independence

Definition: The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots,$ and \mathbf{v}_k are *linearly independent* if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$

Basis of a Subspace

Definition: A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ form a *basis* of a subspace V if

1. $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$
2. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent.

Examples of Bases

- The vector $\mathbf{v} = (1, -1, 1)^T$ is a basis for $\text{null}(A)$.
 - ◆ $\text{null}(A)$ is the subspace of \mathbf{R}^3 with basis \mathbf{v} .
- The vectors $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$ form a basis for $\text{null}(B)$.
 - ◆ $\text{null}(B)$ is the subspace of \mathbf{R}^4 with basis $\{\mathbf{v}, \mathbf{w}\}$.

Basis of a Subspace

Proposition: Let V be a subspace of \mathbf{R}^n .

1. If $V \neq \{0\}$, then V has a basis.
2. Every basis of V has the same number of elements.

Definition: The *dimension* of a subspace V is the number of elements in a basis of V .

Linear Independence?

How do we decide if a set of vectors is **linearly independent**?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}$$

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0} \Leftrightarrow$$

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \mathbf{c} = \mathbf{0} \Leftrightarrow$$

$$\mathbf{c} \in \text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]).$$

- $\mathbf{c} = (-3, 2, 1)^T \in \text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]), \Rightarrow$
 $-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$

- $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 3 \end{pmatrix}$$

- $\text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = \{\mathbf{0}\}$.
- $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Proposition: Suppose that $\mathbf{v}_1, \mathbf{v}_2, \dots,$ and \mathbf{v}_k are vectors in \mathbf{R}^n . Set $V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$.

1. If $\text{null}(V) = \{\mathbf{0}\}$, then $\mathbf{v}_1, \mathbf{v}_2, \dots,$ and \mathbf{v}_k are linearly independent.
2. If $\mathbf{c} = (c_1, c_2, \dots, c_k)^T$ is a nonzero vector in $\text{null}(V)$, then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0},$$

so the vectors are linearly dependent.