Math 211

Lecture #20

Bases of a Subspace

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Subspaces of $\mathbf{R}^n$

Definition: A nonempty subset V of $\mathbf{R}^n$ that has the properties

1. if $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, $\mathbf{x}+\mathbf{y}$ is in V,
2. if a is a scalar, and $\mathbf{x}$ is in V, then $a\mathbf{x}$ is in V,

is called a *subspace* of $\mathbf{R}^n$.

- The nullspace of a matrix is a subspace.
- We are looking for a good way to describe a subspace.

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The Span of a Set of Vectors

In every example we have seen the subspace has been the set of all linear combinations of a few vectors.

Definition: The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ is denoted by

$$\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

Proposition: If $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are all vectors in $\mathbf{R}^n$, then $V = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbf{R}^n$.

Return null(A) null(B)

Linear Dependence in 2- & 3-D

We need a condition that will keep unneeded vectors out of a spanning list. We will work toward a general definition.

- Two vectors are *linearly dependent* if one is a scalar multiple of the other.
- Three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are *linearly dependent* if one is a linear combination of the other two.
  - Example: $\mathbf{v}_1 = (1, 0, 0)^T$, $\mathbf{v}_2 = (0, 1, 0)^T$, and $\mathbf{v}_3 = (1, 2, 0)^T$

$$\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2.$$

  - Notice that $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

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Linear Dependence

- Three vectors are linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
  - Non-trivial means that at least one of the coefficients is not 0.
- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.

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Linear Independence

Definition: The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are *linearly independent* if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$
$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$

Return Three vectors More vectors

Linear Independence?

How do we decide if a set of vectors is linearly independent? Are the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}$$

linearly independent?

We look at linear combinations of the vectors

$$\begin{aligned} & c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \\ \Leftrightarrow\ & [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]\mathbf{c} = \mathbf{0} \quad \text{where} \quad \mathbf{c} = (c_1, c_2, c_3)^T \\ \Leftrightarrow\ & \mathbf{c} \in \text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]). \end{aligned}$$

- $\mathbf{c} = (-3, 2, 1)^T \in \text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3])$,

$$\Rightarrow\ -3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$$

- $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ are linearly dependent.

Another Example

Are the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 3 \end{pmatrix}$$

linearly independent?

- $\text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = \{\mathbf{0}\}$.
- $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ are linearly independent.

Proposition: Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are vectors in $\mathbf{R}^n$. Set $V = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k]$.

1. If $\text{null}(V) = \{\mathbf{0}\}$, then $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are linearly independent.
2. If $\mathbf{c} = (c_1, c_2, \ldots, c_k)^T$ is a nonzero vector in $\text{null}(V)$, then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

so the vectors are linearly dependent.

Basis of a Subspace

Definition: A set of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ form a *basis* of a subspace V if

1. $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$
2. $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are linearly independent.

Examples of Bases

- The vector $\mathbf{v} = (1, -1, 1)^T$ is a basis for $\text{null}(A)$.
  - $\text{null}(A)$ is the subspace of $\mathbf{R}^3$ with basis $\mathbf{v}$.
- The vectors $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$ form a basis for $\text{null}(B)$.
  - $\text{null}(B)$ is the subspace of $\mathbf{R}^4$ with basis $\{\mathbf{v}, \mathbf{w}\}$.

Basis of a Subspace

Proposition: Let V be a subspace of $\mathbf{R}^n$.

1. If $V \neq \{\mathbf{0}\}$, then V has a basis.
2. Every basis of V has the same number of elements.

Definition: The *dimension* of a subspace V is the number of elements in a basis of V.

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Example

Find the nullspace of

$$A = \begin{pmatrix} 3 & -3 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 13 & -13 & 5 & -5 \end{pmatrix}.$$

- $\text{null}(A)$ is the subspace of $\mathbf{R}^4$ with basis $(1, 1, 0, 0)^T$ and $(0, 0, 1, -1)^T$.
- $\text{null}(A)$ has dimension 2.

Example 1

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

The nullspace of A is

$$\text{null}(A) = \{a\mathbf{v} \mid a \in \mathbf{R}\},$$

where $\mathbf{v} = (1,\, -1,\, 1)^T$.

Return

16

Example 2

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

- $\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \mid a,\, b \in \mathbf{R}\}$, where $\mathbf{v} = (1,\, -1,\, 1,\, 0)^T$ and $\mathbf{w} = (0,\, -2,\, 0,\, 1)^T$.
- $\text{null}(B)$ consists of all linear combinations of $\mathbf{v}$ and $\mathbf{w}$.

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