

Math 211

Lecture #22
Systems of ODEs

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Systems of Differential Equations

Example: *SIR* model of the spread of an infectious disease, such as the flu, measles, the common cold, etc.

Assumptions:

- The disease is of short duration and rarely fatal.
- The disease spreads through human contact.
- Recovered individuals are immune.

Return

SIR Model

Divide the population into three subpopulations:

- The susceptible $S(t)$
- The infected $I(t)$
- The recovered $R(t)$
- $N = S + I + R$ is constant.

$$S' = -aSI$$

$$I' = aSI - bI$$

$$R' = bI.$$

- MATLAB & `ppplane5`.

Return

Assumptions

General System in 2D

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

- Example:

$$x' = y$$

$$y' = -x$$

- Solution: $x(t) = \sin t$ and $y(t) = \cos t$
 - ♦ Verify by direct substitution.

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General System in Higher D

$$x'_1 = f_1(t, x_1, x_2, \dots, x_n)$$

$$x'_2 = f_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots = \quad \vdots$$

$$x'_n = f_n(t, x_1, x_2, \dots, x_n)$$

- The *dimension* of a system is the number of unknown functions = the number of equations.
 - ♦ The SIR model has dimension 3 (or 2).

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Planar system

Vector Notation — 2D

- In 2D set $u_1(t) = x(t)$ & $u_2(t) = y(t)$, and

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.$$

- Then in the example

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \Leftrightarrow \mathbf{u}' = \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$$

Return

Vector Notation — Planar System

- For the general case use vector notation and set

$$\mathbf{F}(t, \mathbf{u}) = \begin{pmatrix} f(t, u_1, u_2) \\ g(t, u_1, u_2) \end{pmatrix}.$$

- Then

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t, \mathbf{u})$$

Return

Vector Notation — General

- In higher dimensions, set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

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Vector Notation — SIR Model

For the SIR model set $x_1 = S$, $x_2 = I$, and $x_3 = R$. Then the system can be written

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} -ax_1x_2 \\ ax_1x_2 - bx_2 \\ bx_2 \end{pmatrix} = \mathbf{f}(\mathbf{x}).$$

- This is an *autonomous* system.
 - ♦ The RHS has no explicit dependence on t .

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General case

Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example:

$$\begin{array}{l} x' = y \\ y' = -x \end{array} \quad \text{with} \quad \begin{array}{l} x(0) = 2 \\ y(0) = 13 \end{array}$$

- SIR model: The initial populations in each category must be specified.

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Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.

Return

Example of Reduction

- Third-order equation: $y''' + 2yy' = 3 \cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 3 \cos t - 2x_1x_2$$

- This system is *not* autonomous.

Return

Geometric Interpretation of Solutions

- Parametric plot
 - ♦ Tangent vectors
- Vector fields
- Phase plane
- pp1ane5 for planar autonomous systems.