Math 211

Lecture #22

Systems of ODEs

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Systems of Differential Equations

Example: SIR model of the spread of an infectious disease, such as the flu, measles, the common cold, etc.

Assumptions:

- The disease is of short duration and rarely fatal.
- The disease spreads through human contact.
- Recovered indiviuals are immune.

SIR Model

Divide the population into three subpopulations:

- The susceptible $S(t)$
- The infected $I(t)$
- The recovered $R(t)$
- $N = S + I + R$ is constant.

$$S' = -aSI$$
$$I' = aSI - bI$$
$$R' = bI.$$

- MATLAB & `pplane5`.

General System in 2D

$$x' = f(t, x, y)$$
$$y' = g(t, x, y)$$

- Example:

$$x' = y$$
$$y' = -x$$

- Solution: $x(t) = \sin t$ and $y(t) = \cos t$
  - Verify by direct substitution.

General System in Higher D

$$
\begin{aligned}
x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\
x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\
\vdots &= \qquad \vdots \\
x_n' &= f_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}
$$

- The *dimension* of a system is the number of unknown functions = the number of equations.
  - The SIR model has dimension 3 (or 2).

Vector Notation — 2D

- In 2D set $u_1(t) = x(t)$ & $u_2(t) = y(t)$, and

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.$$

- Then in the example

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \Leftrightarrow \mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$$

Vector Notation — Planar System

- For the general case use vector notation and set

$$\mathbf{F}(t,\mathbf{u}) = \begin{pmatrix} f(t,u_1,u_2) \\ g(t,u_1,u_2) \end{pmatrix}.$$

- Then

$$\begin{aligned} x' &= f(t,x,y) \\ y' &= g(t,x,y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t,\mathbf{u})$$

Vector Notation — General

- In higher dimensions, set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

Vector Notation — SIR Model

For the SIR model set $x_1 = S$, $x_2 = I$, and $x_3 = R$. Then the system can be written

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} -ax_1x_2 \\ ax_1x_2 - bx_2 \\ bx_2 \end{pmatrix} = \mathbf{f}(\mathbf{x}).$$

- This is an *autonomous* system.
  - The RHS has no explicit dependence on t.

Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example:

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \quad \text{with} \quad \begin{aligned} x(0) &= 2 \\ y(0) &= 13 \end{aligned}$$

- SIR model: The initial populations in each category must be specified.

Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.

Example of Reduction

- Third-order equation: $y''' + 2yy' = 3\cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 3\cos t - 2x_1x_2$$

- This system is *not* autonomous.

Geometric Interpretation of Solutions

- Parametric plot
  - Tangent vectors
- Vector fields
- Phase plane
- `pplane5` for planar autonomous systems.