

Math 211

Lecture #23

Qualitative Analysis

October 22, 2001

General System in 2D

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

- Example:

$$x' = y$$

$$y' = -x$$

General System in Higher D

$$x'_1 = f_1(t, x_1, x_2, \dots, x_n)$$

$$x'_2 = f_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots = \quad \quad \quad \vdots$$

$$x'_n = f_n(t, x_1, x_2, \dots, x_n)$$

Vector Notation — General

- Set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each **component** of $\mathbf{x}(t_0)$ must be specified.
- Example

$$\begin{array}{l} x' = y \\ y' = -x \end{array} \quad \text{with} \quad \begin{array}{l} x(0) = 2 \\ y(0) = 13 \end{array}$$

Existence & Uniqueness

General System $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$

- \mathbf{x} in an open set $U \subset \mathbf{R}^n$
- t in an interval $I = (a, b)$

$$R = I \times U = \{(t, \mathbf{x}) \mid t \in I \text{ and } \mathbf{x} \in U\}.$$

Theorem: Suppose that $\mathbf{f}(t, \mathbf{x})$ is continuous in R , and that all first partials of \mathbf{f} are also continuous in R . Then given any $t_0 \in I$ and $\mathbf{x}_0 \in U$ there is a unique solution to the initial value problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

defined on an interval containing t_0 . The solution exists at least until the solution curve $t \rightarrow (t, \mathbf{x}(t))$ leaves R .

Autonomous Systems

System of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

- Look at solution curves $t \rightarrow \mathbf{x}(t) \in \mathbf{R}^n$.
- \mathbf{R}^n is called *phase space*.
 - ◆ If $n = 2$, \mathbf{R}^2 is the *phase plane*.
 - ◆ If $n = 1$, \mathbf{R}^1 is the phase line.

Uniqueness in Phase Space

Two solution curves in phase space for an **autonomous system** cannot meet at a point unless the solution curves coincide.

- If $n = 2$, two solution curves in the phase plane cannot cross, or even touch.
- *If the system is not autonomous, solution curves in the phase plane can cross.*

Equilibrium Points & Solutions

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

- The system is autonomous.
- \mathbf{x}_0 is an *equilibrium point* if $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.
- $\mathbf{x}(t) = \mathbf{x}_0$ is the corresponding *equilibrium solution*.
- In phase space, an equilibrium solution plots as a point.
- Nullclines — where one component of the right-hand side $\mathbf{f}(\mathbf{x})$ vanishes.

Example

$$x' = x^2 - y$$

$$y' = x - xy$$

- **x -nullcline:** $x^2 - y = 0$.
- y -nullcline: $x(1 - y) = 0$.
- 3 equilibrium points: $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

Linear Systems

A system is *linear* if the unknown functions appear linearly in the right-hand sides.

- *Appear linearly* means that there are no products, powers, or higher order functions.
- Examples
 - ◆ SIR is nonlinear.
 - ◆ *Previous example* is linear.

Planar Linear Systems

A planar **linear system** is one of the form

$$x' = a(t)x + b(t)y + f(t)$$

$$y' = c(t)x + d(t)y + g(t)$$

- The coefficients can depend on t .

General Linear Systems

$$x'_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1$$

$$x'_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2$$

$$\vdots = \quad \vdots$$

$$x'_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n$$

- The coefficients can depend on t .

- Set

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^T$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \dots, f_n(t))^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Existence & Uniqueness

Theorem: Suppose the matrix-valued function $A = A(t)$ and the vector-valued function $\mathbf{f}(t)$ are defined and continuous in an interval $I = (\alpha, \beta)$. Then for any t_0 in I and any \mathbf{x}_0 in \mathbf{R}^n , the initial value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution defined *for all t in I* .

Homogeneous Systems

A *homogeneous* system is one of the form

$$\mathbf{x}' = A\mathbf{x}$$

Proposition: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, \dots , and $\mathbf{x}_k(t)$ are solutions to the homogeneous system, and c_1, c_2, \dots , and c_k are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.