

# Math 211

Lecture #24

Linear Systems of ODEs

October 24, 2001

## General Linear Systems

$$x'_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1$$

$$x'_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2$$

$$\vdots = \quad \vdots$$

$$x'_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n$$

- The coefficients can depend on  $t$ .

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- Set

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^T$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \dots, f_n(t))^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes  $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ .

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## Homogeneous Systems

An *homogeneous* system is one of the form

$$\mathbf{x}' = A\mathbf{x}$$

**Proposition:** Suppose that  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$ ,  $\dots$ , and  $\mathbf{x}_k(t)$  are solutions to the homogeneous system  $\mathbf{x}' = A\mathbf{x}$ , and  $c_1, c_2, \dots$ , and  $c_k$  are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

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## Very Important Example

- The system

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

has solutions

$$\mathbf{x}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

- Verify by direct substitution.
- Proposition  $\Rightarrow \mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$  is a solution for any constants  $C_1$  and  $C_2$ .

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- Let  $\mathbf{y}$  be a solution of  $\mathbf{y}' = A\mathbf{y}$ . Can we find  $C_1$  and  $C_2$  so that

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t?$$

- Let's ask a simpler question. Can we find  $C_1$  and  $C_2$  so that

$$\mathbf{y}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)?$$

- Yes, since  $\mathbf{x}_1(0)$  and  $\mathbf{x}_2(0)$  are linearly independent.

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- Uniqueness theorem  $\Rightarrow$

$$\mathbf{y}(t) = C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t) \quad \text{for all } t.$$

- Thus, every solution to  $\mathbf{x}' = A\mathbf{x}$  is a linear combination of  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .
  - ♦ The general solution to the system is a linear combination of the two solutions  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .
- Can we generalize this result?

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Structure

### Key Point in the Argument

- Need to solve the equation

$$\mathbf{y}_0 = C_1 \mathbf{x}_1(0) + C_2 \mathbf{x}_2(0)$$

for any  $\mathbf{y}_0 = \mathbf{y}(0)$ .

- Possible if  $\mathbf{x}_1(0)$  and  $\mathbf{x}_2(0)$  are linearly independent.
- Uniqueness then implies that

$$\mathbf{y}(t) = C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t) \quad \text{for all } t.$$

- We needed  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  to be linearly independent at only one point.

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**Proposition:**  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$ ,  $\dots$ , and  $\mathbf{x}_k(t)$  solutions to the homogeneous system  $\mathbf{x}' = A\mathbf{x}$  on the interval  $I$ .

1. If  $\mathbf{x}_1(t_0)$ ,  $\mathbf{x}_2(t_0)$ ,  $\dots$ , and  $\mathbf{x}_k(t_0)$  are linearly independent for some  $t_0 \in I$ , then they are linearly independent for all  $t \in I$ .
2. If  $\mathbf{x}_1(t_0)$ ,  $\mathbf{x}_2(t_0)$ ,  $\dots$ , and  $\mathbf{x}_k(t_0)$  are linearly dependent for some  $t_0 \in I$ , then they are linearly dependent for all  $t \in I$ .

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Example

## Linear Independence

**Definition:** A set of  $k$  solutions to the linear system  $\mathbf{x}' = A\mathbf{x}$  is *linearly independent* if they are linearly independent at one value of  $t$ .

- Proposition  $\Rightarrow$  the solutions are linearly independent for all values of  $t$ .

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## Structure of the Solution Space

**Theorem:** Suppose that  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$ ,  $\dots$ , and  $\mathbf{x}_n(t)$  are linearly independent solutions to the  $n \times n$  homogeneous system  $\mathbf{x}' = A\mathbf{x}$  on the interval  $I$ . Then every solution is a linear combination of  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$ ,  $\dots$ , and  $\mathbf{x}_n(t)$ .

- That is, if  $\mathbf{x}(t)$  is any solution, then there are constants  $C_1, C_2, \dots$ , and  $C_n$  such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \dots + C_n\mathbf{x}_n(t).$$

- The general solution is a linear combination of  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$ ,  $\dots$ , and  $\mathbf{x}_n(t)$ .

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[Very Important Example](#)

## Solution Strategy

- The obvious strategy for completely solving an  $n \times n$  system is to look for  $n$  linearly independent solutions.

**Definition:** A set of  $n$  linear independent solutions to the  $n \times n$  homogeneous system  $\mathbf{x}' = A\mathbf{x}$  is called a *fundamental set of solutions*.

- We will look for fundamental sets of solutions.

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**Example 1:  $\mathbf{x}' = A\mathbf{x}$** 

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

are a fundamental set of solutions.

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**Example 2:  $\mathbf{x}' = A\mathbf{x}$** 

$$A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$$

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

are a fundamental set of solutions.

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**Linear Systems with Constant Coefficients**

- We will solve homogeneous equations first.
- We will be able to find explicit solutions.
- To motivate what we do, we will start with the easiest case, dimension = 1.
  - ♦ One equation:  $x' = ax$ 
    - ▶  $a$  is a constant.
  - ♦ Solution:  $x(t) = Ce^{at}$
  - ♦ All solutions are exponentials. Can we find exponential solutions to a system of equations?

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## Exponential Solutions to $\mathbf{x}' = A\mathbf{x}$

- Look for solution of the form  $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ , where  $\mathbf{v}$  is a vector with constant entries.
- Substituting we get

$$\begin{aligned}\mathbf{x}' &= \lambda e^{\lambda t}\mathbf{v} \\ A\mathbf{x} &= e^{\lambda t}A\mathbf{v}\end{aligned}$$

- Hence  $\mathbf{x}' = A\mathbf{x} \Leftrightarrow A\mathbf{v} = \lambda\mathbf{v}$
- If  $A\mathbf{v} = \lambda\mathbf{v}$  then  $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$  is a solution.
- Can we find  $\lambda$  and  $\mathbf{v}$  such that  $A\mathbf{v} = \lambda\mathbf{v}$ ?

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Dimension = 1

## Eigenvalues & Eigenvectors

**Definition:**  $\lambda$  is an *eigenvalue* of  $A$  if there is a nonzero vector  $\mathbf{v}$  such that  $A\mathbf{v} = \lambda\mathbf{v}$ . If  $\lambda$  is an eigenvalue of  $A$ , then any vector  $\mathbf{v}$  such that  $A\mathbf{v} = \lambda\mathbf{v}$  is called an *eigenvector associated with  $\lambda$* .

- If  $\lambda$  an eigenvalue of  $A$ , and  $\mathbf{v}$  is an associated nonzero eigenvector, then  $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$  is a solution to  $\mathbf{x}' = A\mathbf{x}$ .
- How do we find eigenvalues and eigenvectors?

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Exponential solution

## Finding Eigenvalues

$\lambda$  is an eigenvalue of  $A$

$\Leftrightarrow$  there is a vector  $\mathbf{v} \neq \mathbf{0}$  such that  $A\mathbf{v} = \lambda\mathbf{v}$ .

$\Leftrightarrow \mathbf{v} \neq \mathbf{0}$  and  $\mathbf{0} = A\mathbf{v} - \lambda\mathbf{v}$

$$= A\mathbf{v} - \lambda I\mathbf{v}$$

$$= (A - \lambda I)\mathbf{v}$$

$\Leftrightarrow A - \lambda I$  has a nontrivial nullspace.

$\Leftrightarrow \det(A - \lambda I) = 0$ .

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### Example

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -4 - \lambda & 2 \\ -3 & 1 - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (-4 - \lambda)(1 - \lambda) + 6$$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda + 1)(\lambda + 2)$$

- $A$  has eigenvalues  $\lambda_1 = -1$  and  $\lambda_2 = -2$ .

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Finding eigenvalues

$$\det(A - \lambda I)$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix}$$

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Finding eigenvalues

- If  $A$  is an  $n \times n$  matrix  $p(\lambda) = \det(A - \lambda I)$  is a polynomial of degree  $n$ .

**Definition:** The *characteristic polynomial* of the  $n \times n$  matrix  $A$  is

$$p(\lambda) = \det(A - \lambda I).$$

The *characteristic equation* is  $p(\lambda) = 0$ .

- The eigenvalues of  $A$  are the roots of the characteristic equation.

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- Usually  $p(\lambda) = \det(A - \lambda I) = 0$  has  $n$  roots. Usually  $A$  has  $n$  eigenvalues.
- Each eigenvalue  $\lambda$  has by definition an associated nonzero eigenvector  $\mathbf{v}$ .
- Each nonzero eigenvector leads to a solution,  $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$ .
- There is at least one solution for every eigenvalue, so we expect  $n$  different solutions.
  - ♦ Are they linearly independent?

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Example

### Finding Eigenvectors

- $\mathbf{v}$  is an eigenvector associated with the eigenvalue  $\lambda$  if

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ \Leftrightarrow (A - \lambda I)\mathbf{v} &= \mathbf{0} \\ \Leftrightarrow \mathbf{v} \in \text{null}(A - \lambda I) \end{aligned}$$

- The set of all eigenvectors associated to the eigenvalue  $\lambda$  is equal to the nullspace of  $A - \lambda I$ . It is a subspace of  $\mathbf{R}^n$  called the *eigenspace* of  $\lambda$ .

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Finding eigenvalues

### Example

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

- $A$  has eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -2.$$

- Look for associated eigenvectors. Each leads to a solution to  $\mathbf{x}' = A\mathbf{x}$ .

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$$\lambda_1 = -1$$

$$A - \lambda_1 I = \begin{pmatrix} -4+1 & 2 \\ -3 & 1+1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ is an eigenvector}$$

$$\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ is a solution.}$$

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$$\lambda_2 = -2$$

$$A - \lambda_2 I = \begin{pmatrix} -4+2 & 2 \\ -3 & 1+2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector}$$

$$\mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2 = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is a solution.}$$

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$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

- The system has solutions

$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- $\mathbf{x}_1(0) = \mathbf{v}_1$  and  $\mathbf{x}_2(0) = \mathbf{v}_2$  are linearly independent.
- $\mathbf{x}_1$  and  $\mathbf{x}_2$  form a fundamental set of solutions.

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$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

- The general solution is the set of all linear combinations:

$$\begin{aligned} \mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \\ &= C_1e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2C_1e^{-t} + C_2e^{-2t} \\ 3C_1e^{-t} + C_2e^{-2t} \end{pmatrix} \end{aligned}$$

### Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of  $A$ 
  - ♦ the roots of  $\det(A - \lambda I) = 0$
- For each eigenvalue  $\lambda$  find the eigenspace
  - ♦  $= \text{null}(A - \lambda I)$
- If  $\lambda$  is an eigenvalue and  $\mathbf{v}$  is an associated nonzero eigenvector,  $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$  is a solution.
- Show that  $n$  of these are linearly independent, if we can.

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