

Math 211

Lecture #24

Linear Systems of ODEs

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General Linear Systems

$$x'_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1$$

$$x'_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2$$

$$\vdots = \quad \vdots$$

$$x'_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n$$

- The coefficients can depend on t .

- Set

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^T$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \dots, f_n(t))^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Homogeneous Systems

An *homogeneous* system is one of the form

$$\mathbf{x}' = A\mathbf{x}$$

Proposition: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, \dots , and $\mathbf{x}_k(t)$ are solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$, and c_1 , c_2 , \dots , and c_k are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

Very Important Example

- The system

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

has solutions

$$\mathbf{x}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

- ♦ Verify by direct substitution.
- **Proposition** $\Rightarrow \mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$ is a solution for any constants C_1 and C_2 .

- Let \mathbf{y} be a solution of $\mathbf{y}' = A\mathbf{y}$. Can we find C_1 and C_2 so that

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t?$$

- Let's ask a simpler question. Can we find C_1 and C_2 so that

$$\mathbf{y}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)?$$

- ♦ Yes, since $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.

- Uniqueness theorem \Rightarrow

$$\mathbf{y}(t) = C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t) \quad \text{for all } t.$$

- Thus, every solution to $\mathbf{x}' = A\mathbf{x}$ is a linear combination of \mathbf{x}_1 and \mathbf{x}_2 .
 - ♦ The general solution to the system is a linear combination of the two solutions \mathbf{x}_1 and \mathbf{x}_2 .
- Can we generalize this result?

Key Point in the Argument

- Need to solve the equation

$$\mathbf{y}_0 = C_1 \mathbf{x}_1(0) + C_2 \mathbf{x}_2(0)$$

for any $\mathbf{y}_0 = \mathbf{y}(0)$.

- Possible if $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.
- Uniqueness then implies that

$$\mathbf{y}(t) = C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t) \quad \text{for all } t \quad .$$

- We needed $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to be linearly independent at only one point.

Proposition: $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots,$ and $\mathbf{x}_k(t)$ solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I .

1. If $\mathbf{x}_1(t_0), \mathbf{x}_2(t_0), \dots,$ and $\mathbf{x}_k(t_0)$ are linearly independent for some $t_0 \in I$, then they are linearly independent for all $t \in I$.
2. If $\mathbf{x}_1(t_0), \mathbf{x}_2(t_0), \dots,$ and $\mathbf{x}_k(t_0)$ are linearly dependent for some $t_0 \in I$, then they are linearly dependent for all $t \in I$.

Linear Independence

Definition: A set of k solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is *linearly independent* if they are linearly independent at one value of t .

- **Proposition** \Rightarrow the solutions are linearly independent for all values of t .

Structure of the Solution Space

Theorem: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, \dots , and $\mathbf{x}_n(t)$ are **linearly independent** solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I . Then every solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, \dots , and $\mathbf{x}_n(t)$.

- That is, if $\mathbf{x}(t)$ is any solution, then there are constants C_1, C_2, \dots , and C_n such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \dots + C_n\mathbf{x}_n(t).$$

- The general solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, \dots , and $\mathbf{x}_n(t)$.

Solution Strategy

- The obvious **strategy** for completely solving an $n \times n$ system is to look for n linearly independent solutions.

Definition: A set of n linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a *fundamental set of solutions*.

- We will look for fundamental sets of solutions.

Example 1: $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

are a **fundamental set** of solutions.

Example 2: $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$$

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

are a **fundamental set** of solutions.

Linear Systems with Constant Coefficients

- We will solve **homogeneous equations** first.
- We will be able to find explicit solutions.
- To motivate what we do, we will start with the easiest case, dimension = 1.
 - ◆ One equation: $x' = ax$
 - ▶ a is a constant.
 - ◆ Solution: $x(t) = Ce^{at}$
 - ◆ All solutions are exponentials. Can we find exponential solutions to a system of equations?

Exponential Solutions to $\mathbf{x}' = A\mathbf{x}$

- Look for solution of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, where \mathbf{v} is a vector with constant entries.
- Substituting we get

$$\mathbf{x}' = \lambda e^{\lambda t}\mathbf{v}$$

$$A\mathbf{x} = e^{\lambda t}A\mathbf{v}$$

- Hence $\mathbf{x}' = A\mathbf{x} \iff A\mathbf{v} = \lambda\mathbf{v}$
- If $A\mathbf{v} = \lambda\mathbf{v}$ then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Can we find λ and \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$?

Eigenvalues & Eigenvectors

Definition: λ is an *eigenvalue* of A if there is a nonzero vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$. If λ is an eigenvalue of A , then any vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with λ* .

- If λ an eigenvalue of A , and \mathbf{v} is an associated nonzero eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.
- How do we find eigenvalues and eigenvectors?

Finding Eigenvalues

λ is an **eigenvalue** of A

\Leftrightarrow there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

$$\begin{aligned}\Leftrightarrow \mathbf{v} \neq \mathbf{0} \text{ and } \mathbf{0} &= A\mathbf{v} - \lambda\mathbf{v} \\ &= A\mathbf{v} - \lambda I\mathbf{v} \\ &= (A - \lambda I)\mathbf{v}\end{aligned}$$

$\Leftrightarrow A - \lambda I$ has a nontrivial nullspace.

$\Leftrightarrow \det(A - \lambda I) = 0$.

Example

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -4 - \lambda & 2 \\ -3 & 1 - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (-4 - \lambda)(1 - \lambda) + 6$$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda + 1)(\lambda + 2)$$

- A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

$$\det(A - \lambda I)$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix}$$

- If A is an $n \times n$ matrix $p(\lambda) = \det(A - \lambda I)$ is a **polynomial** of degree n .

Definition: The *characteristic polynomial* of the $n \times n$ matrix A is

$$p(\lambda) = \det(A - \lambda I).$$

The *characteristic equation* is $p(\lambda) = 0$.

- The **eigenvalues** of A are the roots of the characteristic equation.

- Usually $p(\lambda) = \det(A - \lambda I) = 0$ has n roots. Usually A has n eigenvalues.
- Each eigenvalue λ has by **definition** an associated nonzero eigenvector \mathbf{v} .
- Each nonzero eigenvector leads to a **solution**, $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$.
- There is at least one solution for every eigenvalue, so we expect n different solutions.
 - ◆ Are they linearly **independent**?

Finding Eigenvectors

- \mathbf{v} is an **eigenvector** associated with the eigenvalue λ if

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$\Leftrightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$$

$$\Leftrightarrow \mathbf{v} \in \text{null}(A - \lambda I)$$

- The set of all eigenvectors associated to the eigenvalue λ is equal to the nullspace of $A - \lambda I$. It is a subspace of \mathbf{R}^n called the **eigenspace** of λ .

Example

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

- A has eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -2.$$

- Look for associated eigenvectors. Each leads to a **solution** to $\mathbf{x}' = A\mathbf{x}$.

$$\lambda_1 = -1$$

$$A - \lambda_1 I = \begin{pmatrix} -4 + 1 & 2 \\ -3 & 1 + 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ is an eigenvector}$$

$$\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ is a solution.}$$

$$\lambda_2 = -2$$

$$A - \lambda_2 I = \begin{pmatrix} -4 + 2 & 2 \\ -3 & 1 + 2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector}$$

$$\mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2 = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is a solution.}$$

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

- The system has solutions

$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- $\mathbf{x}_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly **independent**.
- \mathbf{x}_1 and \mathbf{x}_2 form a **fundamental** set of solutions.

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

- The **general solution** is the set of all linear combinations:

$$\begin{aligned} \mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \\ &= C_1e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2C_1e^{-t} + C_2e^{-2t} \\ 3C_1e^{-t} + C_2e^{-2t} \end{pmatrix} \end{aligned}$$

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A
 - ◆ the roots of $\det(A - \lambda I) = 0$
- For each eigenvalue λ find the eigenspace
 - ◆ $= \text{null}(A - \lambda I)$
- If λ is an eigenvalue and \mathbf{v} is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent, if we can.