Math 211

Lecture #26

Solutions of a Planar System

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Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A
  - the roots of $p(\lambda) = \det(A - \lambda I) = 0$
- For each eigenvalue λ find the eigenspace
  - $= \mathrm{null}(A - \lambda I)$
- If λ is an eigenvalue and $\mathbf{v}$ is an associated eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Hope that n of these are linearly independent.

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

where $T = \operatorname{tr} A$ and $D = \det A$

- The eigenvalues of A are the roots of $p(\lambda) = \lambda^2 - T\lambda + D$,

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- Three cases:
  - 2 distinct real roots if $T^2 - 4D > 0$
  - 2 complex conjugate roots if $T^2 - 4D < 0$
  - Double real root if $T^2 - 4D = 0$

Complex Eigenvalues

- If the discriminant $T^2 - 4D < 0$ we have complex eigenvalues

$$\lambda = \frac{T + i\sqrt{4D - T^2}}{2}, \quad \overline{\lambda} = \frac{T - i\sqrt{4D - T^2}}{2}$$

- Example: $\begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$.
  - Characteristic polynomial: $p(\lambda) = \lambda^2 - 2\lambda + 5$.
  - Eigenvalues: $\lambda = 1 + 2i$ and $\overline{\lambda} = 1 - 2i$
  - Eigenvector: $\mathbf{w} = \begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$

Complex Eigenpairs

A a real matrix

- Suppose that λ is a complex eigenvalue with associated eigenvector $\mathbf{w}$. Then $A\mathbf{w} = \lambda\mathbf{w}$.
- Conjugating we get

$$\overline{A\mathbf{w}} = \overline{A}\,\overline{\mathbf{w}} = A\overline{\mathbf{w}}$$

$$\overline{\lambda\mathbf{w}} = \overline{\lambda}\,\overline{\mathbf{w}}$$

- $A\mathbf{w} = \lambda\mathbf{w} \;\Rightarrow\; \overline{A\mathbf{w}} = \overline{\lambda\mathbf{w}} \;\Rightarrow\; A\overline{\mathbf{w}} = \overline{\lambda}\,\overline{\mathbf{w}}$
- $\Rightarrow$ $\overline{\lambda}$ is an eigenvalue of A with associated eigenvector $\overline{\mathbf{w}}$

- Thus complex eigenvalues come in conjugate pairs λ and $\overline{\lambda}$.
- The associated eigenvectors also come in conjugate pairs $\mathbf{w}$ and $\overline{\mathbf{w}}$.
- $\lambda \neq \overline{\lambda} \quad \Rightarrow \quad \mathbf{w}$ and $\overline{\mathbf{w}}$ are linearly independent.
- We get complex exponential solutions

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} \quad \text{and} \quad \overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}.$$

- $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent complex valued solutions to $\mathbf{x}' = A\mathbf{x}$.

$$\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t) \quad \& \quad \overline{\mathbf{z}}(t) = \mathbf{x}(t) - i\mathbf{y}(t)$$

$$\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) + \overline{\mathbf{z}}(t)}{2}$$

$$\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) - \overline{\mathbf{z}}(t)}{2i}$$

- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are real valued solutions.
- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are linearly independent.

Solutions in Our Example

- Complex Solutions

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} = e^{(1+2i)t}\begin{pmatrix} 3-i \\ 1 \end{pmatrix}$$

$$\overline{\mathbf{z}}(t) = e^{\overline{\lambda}t}\overline{\mathbf{w}} = e^{(1-2i)t}\begin{pmatrix} 3+i \\ 1 \end{pmatrix}$$

- Real Solutions

$$\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t)) = e^{t}\begin{pmatrix} 3\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t)) = e^{t}\begin{pmatrix} 3\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$$

Initial Value Problem

Solve

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$$

with the intial condition

$$\mathbf{x}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Initial Value Problem

Solution is

$$\mathbf{u}(t) = 3e^t \begin{pmatrix} 3\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} + 4e^t \begin{pmatrix} 3\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} = e^t \begin{pmatrix} 5\cos 2t + 15\sin 2t \\ 3\cos 2t + 4\sin 2t \end{pmatrix}$$

Summary — Complex Eigenvalues

Suppose A is a real 2×2 matrix with

- complex conjugate eigenvalues λ and $\overline{\lambda}$, and
- associated nonzero eigenvectors $\mathbf{w}$ and $\overline{\mathbf{w}}$.

Then

- $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$ and $\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}$ form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t))$ form a real valued fundamental set of solutions.

Examples

$$\mathbf{x}' = A\mathbf{x}$$

where

- $A = \begin{pmatrix} 7 & 30 \\ -3 & -11 \end{pmatrix}$
- $A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$

Complex Matrices

Matrices (or vectors) with complex entries inherit many of the properties of complex numbers.

- $M = A + iB$ where $A = \mathrm{Re}M$ and $B = \mathrm{Im}M$ are real matrices.
- $\overline{M} = A - iB; \quad M = \overline{M} \Leftrightarrow M$ is real.
- $\mathrm{Re}M = \frac{1}{2}(M + \overline{M}); \quad \mathrm{Im}M = \frac{1}{2i}(M - \overline{M})$
- $\overline{M + N} = \overline{M} + \overline{N}$
- $\overline{M\mathbf{z}} = \overline{M}\overline{\mathbf{z}}$

Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

Proposition: Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of the $n \times n$ matrix A, and that $\mathbf{v}_1 \neq 0$ and $\mathbf{v}_2 \neq 0$ are eigenvectors associated with λ_1 and λ_2, respectively. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.