

Math 211

Lecture #27

Planar Systems

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Procedure to Solve $\mathbf{x}' \equiv A\mathbf{x}$

- Find the eigenvalues of A
 - ◆ the roots of $p(\lambda) = \det(A - \lambda I) = 0$
- For each eigenvalue λ find the eigenspace
 - ◆ $= \text{null}(A - \lambda I)$
- If λ is an eigenvalue and \mathbf{v} is an associated eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Hope that n of these are linearly independent.

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

where $T = \text{tr } A$ and $D = \det A$

- The **eigenvalues** of A are the roots of $p(\lambda) = \lambda^2 - T\lambda + D$,

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- Three cases:
 - ♦ 2 distinct real roots if $T^2 - 4D > 0$
 - ♦ 2 complex conjugate roots if $T^2 - 4D < 0$
 - ♦ Double real root if $T^2 - 4D = 0$

Complex Eigenpairs

A a real matrix

- Complex eigenvalues come in conjugate pairs λ and $\bar{\lambda}$.
- The associated eigenvectors also come in conjugate pairs \mathbf{w} and $\bar{\mathbf{w}}$.
- $\lambda \neq \bar{\lambda} \Rightarrow \mathbf{w}$ and $\bar{\mathbf{w}}$ are linearly independent.
- We get complex exponential solutions

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{w} \quad \text{and} \quad \bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}}.$$

- \mathbf{z} and $\bar{\mathbf{z}}$ are linearly independent complex valued solutions to $\mathbf{x}' = A\mathbf{x}$.

Complex Eigenpairs – Real Solutions

$$\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t) \quad \& \quad \bar{\mathbf{z}}(t) = \mathbf{x}(t) - i\mathbf{y}(t)$$

$$\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) + \bar{\mathbf{z}}(t)}{2}$$

$$\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) - \bar{\mathbf{z}}(t)}{2i}$$

- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are real valued solutions.
- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are linearly independent.

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}.$$

- Complex Solutions

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{w} = e^{(1+2i)t} \begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$$

$$\bar{\mathbf{z}}(t) = e^{\bar{\lambda}t} \bar{\mathbf{w}} = e^{(1-2i)t} \begin{pmatrix} 3 + i \\ 1 \end{pmatrix}$$

- ♦ \mathbf{z} and $\bar{\mathbf{z}}$ form a complex valued fundamental set of solutions.

- Real Solutions

$$\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3 \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3 \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$$

- ♦ x and y form a real-valued fundamental set of solutions.

Initial Value Problem

Solve $\mathbf{x}' = A\mathbf{x}$ & $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, where $A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$.

The solution is

$$\begin{aligned} \mathbf{u}(t) &= 3e^t \begin{pmatrix} 3 \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} \\ &\quad + 4e^t \begin{pmatrix} 3 \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} \\ &= e^t \begin{pmatrix} 5 \cos 2t + 15 \sin 2t \\ 3 \cos 2t + 4 \sin 2t \end{pmatrix} \end{aligned}$$

Summary — Complex Eigenvalues

Suppose A is a real 2×2 matrix with

- complex conjugate eigenvalues λ and $\bar{\lambda}$, and
- associated nonzero eigenvectors \mathbf{w} and $\bar{\mathbf{w}}$.

Then

- $\mathbf{z}(t) = e^{\lambda t} \mathbf{w}$ and $\bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}}$ form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t))$ form a real valued fundamental set of solutions.

Examples

$$\mathbf{x}' = A\mathbf{x}$$

where

- $A = \begin{pmatrix} 7 & 30 \\ -3 & -11 \end{pmatrix}$

- $A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$

Double Real Root

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{T}{2}.$$

- $T^2 - 4D = 0$
- First possibility:
 - ♦ Eigenspace has dimension 2: $\Leftrightarrow A = \lambda I$.
 - ♦ Every vector is an eigenvector. Every solution has the form

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{v}.$$

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}$$

- $p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$; $\lambda = -2$
- $A - \lambda I = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}$; $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$
- Eigenspace has dimension 1, with basis \mathbf{v}_1 .
- One solution:

$$\mathbf{x}_1(t) = e^{\lambda t} \mathbf{v}_1 = e^{-2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Double Real Root

- Suppose the eigenspace has **dimension** 1.
 - ◆ Spanned by the eigenvector $\mathbf{v}_1 \neq 0$
 - ◆ Standard procedure gives only one solution,

$$\mathbf{x}_1(t) = e^{\lambda t} \mathbf{v}_1.$$

- Look for a second solution of the form

$$\mathbf{x}_2(t) = e^{\lambda t} [\mathbf{v}_2 + t\mathbf{v}_1]$$

- Second solution of the form

$$\mathbf{x}_2(t) = e^{\lambda t} [\mathbf{v}_2 + t\mathbf{v}_1]$$

$$\mathbf{x}'_2 = e^{\lambda t} [(\mathbf{v}_1 + \lambda\mathbf{v}_2) + \lambda t\mathbf{v}_1]$$

$$A\mathbf{x}_2 = e^{\lambda t} [A\mathbf{v}_2 + tA\mathbf{v}_1]$$

- $\mathbf{x}'_2 = A\mathbf{x}_2 \Leftrightarrow$

$$A\mathbf{v}_1 = \lambda\mathbf{v}_1 \quad \text{and} \quad A\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2.$$

- Need:
 - ♦ \mathbf{v}_1 to be an eigenvector.
 - ♦ $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$.

Procedure in Degenerate Planar Case

- Find the (only) eigenvalue λ_1 .
- Find an eigenvector $\mathbf{v}_1 \neq \mathbf{0}$.
- **Find** \mathbf{v}_2 with $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$.
 - ◆ Start with any vector \mathbf{w} not a multiple of \mathbf{v}_1
 - ◆ $(A - \lambda I)\mathbf{w} = a\mathbf{v}_1$ with $a \neq 0$.
 - ◆ Set $\mathbf{v}_2 = \frac{1}{a}\mathbf{w}$. \mathbf{v}_2 is not a multiple of \mathbf{v}_1 .
- $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1]$ form a fundamental set of solutions.

Example (cont.)

- Start with $\mathbf{w} = (1, 0)^T$.

- $\mathbf{v}_2 = -\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

- Fundamental set of solutions:

$$\mathbf{x}_1(t) = e^{\lambda t} \mathbf{v}_1 = e^{-2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{x}_2(t) &= e^{\lambda t} [\mathbf{v}_2 + t\mathbf{v}_1] \\ &= e^{-2t} \begin{pmatrix} -1 - 3t \\ t \end{pmatrix}. \end{aligned}$$

Examples

Solve $\mathbf{x}' = A\mathbf{x}$, where

-

$$A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

-

$$A = \begin{pmatrix} 0 & 9 \\ -1 & -6 \end{pmatrix}$$

Planar System $\mathbf{x}' \equiv A\mathbf{x}$

- Equilibrium points for the system
 - ◆ Set of equilibrium points equals $\text{null}(A)$.
 - ◆ A nonsingular \Rightarrow only equilibrium point is $\mathbf{0}$.
- Can we list the types of all possible equilibrium points for planar linear systems?
 - ◆ Six most important cases.
 - ◆ Look at solution curves in the phase plane.

Exponential Solutions

$$\mathbf{x}(t) = Ce^{\lambda t} \mathbf{v}$$

- The solution curve is a straight half-line through $C\mathbf{v}$. Sometimes called *half-line* solutions.
- If $\lambda > 0$ the solution starts at $\mathbf{0}$ for $t = -\infty$, and tends to ∞ as $t \rightarrow \infty$. *Unstable solution*
- If $\lambda < 0$ the solution starts at ∞ for $t = -\infty$, and tends to $\mathbf{0}$ as $t \rightarrow \infty$. *Stable solution*

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- Eigenvectors \mathbf{v}_1 and \mathbf{v}_2 . General solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$$

Saddle Point

- $\lambda_1 < 0 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$
- Two stable exponential solutions ($C_2 = 0$)
- Two unstable exponential solutions ($C_1 = 0$).
- $C_1 \neq 0$ and $C_2 \neq 0$.
 - ♦ As $t \rightarrow \infty$, $\mathbf{x}(t) \rightarrow \infty$, approaching the half-line through $C_2 \mathbf{v}_2$.
 - ♦ As $t \rightarrow -\infty$, $\mathbf{x}(t) \rightarrow \infty$, approaching the half-line through $C_2 \mathbf{v}_1$.

Nodal Sink

- $\lambda_1 < \lambda_2 < 0$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$
- Four stable exponential solutions.
- All solutions $\rightarrow \mathbf{0}$ as $t \rightarrow \infty$. (Stable)
 - ♦ Tangent to $C_2 \mathbf{v}_2$ if $C_2 \neq 0$.
- All solutions $\rightarrow \infty$ as $t \rightarrow -\infty$.
 - ♦ \parallel to the half line through $C_1 \mathbf{v}_1$ if $C_1 \neq 0$.

Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$
- Four unstable exponential solutions.
- All solutions $\rightarrow \mathbf{0}$ as $t \rightarrow -\infty$.
 - ♦ Tangent to $C_1 \mathbf{v}_1$ if $C_1 \neq 0$.
- All solutions $\rightarrow \infty$ as $t \rightarrow \infty$. (Unstable)
 - ♦ \parallel to the half line through $C_2 \mathbf{v}_2$ if $C_2 \neq 0$.

Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

Proposition: Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of the $n \times n$ matrix A , and that $\mathbf{v}_1 \neq 0$ and $\mathbf{v}_2 \neq 0$ are eigenvectors associated with λ_1 and λ_2 , respectively. Then \mathbf{v}_1 and \mathbf{v}_2 are linearly independent.