Math 211

Lecture #28

Phase Plane Portraits

November 2, 2001

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A
  - the roots of $p(\lambda) = \det(A - \lambda I) = 0$
- For each eigenvalue λ find the eigenspace
  - $= \text{null}(A - \lambda I)$
- If λ is an eigenvalue and $\mathbf{v}$ is an associated eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Hope that n of these are linearly independent.

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

where $T = \operatorname{tr} A$ and $D = \det A$

- The eigenvalues of A are the roots of $p(\lambda) = \lambda^2 - T\lambda + D$,

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- Three cases:
  - 2 distinct real roots if $T^2 - 4D > 0$
  - 2 complex conjugate roots if $T^2 - 4D < 0$
  - Double real root if $T^2 - 4D = 0$

Procedure in Degenerate Planar Case

- Find the (only) eigenvalue λ_1.
- Find an eigenvector $\mathbf{v}_1 \neq \mathbf{0}$.
- Find $\mathbf{v}_2$ with $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$.
  - Start with any vector $\mathbf{w}$ not a multiple of $\mathbf{v}_1$
  - Then $(A - \lambda I)\mathbf{w} = a\mathbf{v}_1$ with $a \neq 0$.
  - Set $\mathbf{v}_2 = \frac{1}{a}\mathbf{w}$. $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$.
- $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1]$ form a fundamental set of solutions.

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}$$

- $p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2; \quad \lambda = -2$
- $A - \lambda I = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}; \quad \mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$
- Eigenspace has dimension 1, with basis $\mathbf{v}_1$.
- One exponential solution:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 = e^{-2t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

- Second solution
  - Start with $\mathbf{w} = (1, 0)^T$.
  - $\mathbf{v}_2 = -\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$
- Fundamental set of solutions:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 = e^{-2t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{x}_2(t) &= e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1] \\ &= e^{-2t}\begin{pmatrix} -1 - 3t \\ t \end{pmatrix}. \end{aligned}$$

Examples

Solve $\mathbf{x}' = A\mathbf{x}$, where

- $$A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

- $$A = \begin{pmatrix} 0 & 9 \\ -1 & -6 \end{pmatrix}$$

Planar System $\mathbf{x}' = A\mathbf{x}$

- Equilibrium points for the system
  - Set of equilibrium points equals $\mathrm{null}(A)$.
  - A nonsingular $\Rightarrow$ only equilibrium point is $\mathbf{0}$.
- Can we list the types of all possible equilibrium points for planar linear systems?
  - We will do the six most important cases.
    - The other cases are Project #3.
  - Look at solution curves in the phase plane.

Exponential Solutions

$$\mathbf{x}(t) = Ce^{\lambda t}\mathbf{v}$$

- The solution curve is a straight half-line through $C\mathbf{v}$. Sometimes called *half-line* solutions.
- If $\lambda > 0$ the solution starts at $\mathbf{0}$ for $t = -\infty$, and tends to ∞ as $t \to \infty$. *Unstable solution*
- If $\lambda < 0$ the solution starts at ∞ for $t = -\infty$, and tends to $\mathbf{0}$ as $t \to \infty$. *Stable solution*

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. General solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

Saddle Point

- $\lambda_1 < 0 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Two stable exponential solutions $(C_2 = 0)$
- Two unstable exponential solutions $(C_1 = 0)$.
- $C_1 \neq 0$ and $C_2 \neq 0$.
  - As $t \to \infty$, $\mathbf{x}(t) \to \infty$, approaching the half-line through $C_2\mathbf{v}_2$.
  - As $t \to -\infty$, $\mathbf{x}(t) \to \infty$, approaching the half-line through $C_2\mathbf{v}_1$.

Nodal Sink

- $\lambda_1 < \lambda_2 < 0$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Four stable exponential solutions.
- All solutions $\to \mathbf{0}$ as $t \to \infty$. (Stable)
  - Tangent to $C_2\mathbf{v}_2$ if $C_2 \neq 0$.
- All solutions $\to \infty$ as $t \to -\infty$.
  - $\|$ to the half line through $C_1\mathbf{v}_1$ if $C_1 \neq 0$.

Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Four unstable exponential solutions.
- All solutions $\to \mathbf{0}$ as $t \to -\infty$.
  - Tangent to $C_1\mathbf{v}_1$ if $C_1 \neq 0$.
- All solutions $\to \infty$ as $t \to \infty$. (Unstable)
  - $\|$ to the half line through $C_2\mathbf{v}_2$ if $C_2 \neq 0$.

Complex Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D < 0$

$$\lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta.$$

- Eigenvector $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ associated to λ.
- Complex solutions

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} = e^{t(\alpha+i\beta)}[\mathbf{v}_1 + i\mathbf{v}_2]$$

$$\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}} = e^{t(\alpha-i\beta)}[\mathbf{v}_1 - i\mathbf{v}_2]$$

- Real solutions

$$\mathbf{x}_1(t) = \mathrm{Re}(\mathbf{z}(t)) = e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2]$$

$$\mathbf{x}_2(t) = \mathrm{Im}(\mathbf{z}(t)) = e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- General solution

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] + C_2 e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

Center

- $\alpha = \mathrm{Re}(\lambda) = 0$
- General real solution

$$\mathbf{x}(t) = C_1[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] + C_2[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- Every solution is periodic with period $T = 2\pi/\beta$.
- All solution curves are ellipses.

Spiral Sink

- $\alpha = \mathrm{Re}(\lambda) < 0$
- General real solution

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] + C_2 e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into $\mathbf{0}$ as $t \to \infty$.

Spiral Source

- $\alpha = \mathrm{Re}(\lambda) > 0$
- General real solution

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] + C_2 e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into $\mathbf{0}$ as $t \to -\infty$.

Planar Systems

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- Char. polynomial $p(\lambda) = \lambda^2 - T\lambda + D$.
- Eigenvalues

$$\lambda_1,\ \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- $\lambda_1 \,\&\, \lambda_2$ are the roots of $p(\lambda)$, so

$$\begin{aligned} p(\lambda) &= \lambda^2 - T\lambda + D \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \end{aligned}$$

- $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$.
- Duality between $(\lambda_1,\ \lambda_2)$ and (T, D).
- Represent systems by location of (T, D) in the TD-plane.

Trace-Determinant Plane

- $T^2 - 4D > 0$
  - $\Rightarrow$ distinct real eigenvalues $\lambda_1 \,\&\, \lambda_2$
  - $D = \lambda_1 \lambda_2 < 0 \Rightarrow$ Saddle point.
  - $D = \lambda_1 \lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
    - $T = \lambda_1 + \lambda_2 > 0 \Rightarrow$ Nodal source.
    - $T = \lambda_1 + \lambda_2 < 0 \Rightarrow$ Nodal sink.

- $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues

$$\lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta.$$

  - $T = \lambda + \overline{\lambda} = 2\alpha > 0 \Rightarrow$ Spiral source.
  - $T = \lambda + \overline{\lambda} = 2\alpha < 0 \Rightarrow$ Spiral sink.
  - $T = \lambda + \overline{\lambda} = 2\alpha = 0 \Rightarrow$ Center.

Types of Equilibrium Points

- *Generic* types
  - Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.
- *Nongeneric* types
  - Center and many others. Occupy pieces of the boundaries between the generic types.

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