

Math 211

Lecture #28

Phase Plane Portraits

Systems of Higher Dimension

November 5, 2001

Planar System $\mathbf{x}' = A\mathbf{x}$

- Equilibrium points for the system
 - ◆ Set of equilibrium points equals $\text{null}(A)$.
 - ◆ A nonsingular \Rightarrow only equilibrium point is $\mathbf{0}$.
- Can we list the types of all possible equilibrium points for planar linear systems?
 - ◆ We will do the six most important cases.
 - ▶ The other cases are Project #3.
 - ◆ Look at solution curves in the phase plane.

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- Eigenvectors \mathbf{v}_1 and \mathbf{v}_2 . General solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$$

- $\lambda_1 < 0 < \lambda_2$ Saddle point.
- $\lambda_1 < \lambda_2 < 0$ Nodal sink.
- $0 < \lambda_1 < \lambda_2$ Nodal source.

Complex Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D < 0$

$$\lambda = \alpha + i\beta \quad \text{and} \quad \bar{\lambda} = \alpha - i\beta.$$

- Eigenvector $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ associated to λ .
- Complex solutions

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{w} = e^{t(\alpha+i\beta)} [\mathbf{v}_1 + i\mathbf{v}_2]$$

$$\bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}} = e^{t(\alpha-i\beta)} [\mathbf{v}_1 - i\mathbf{v}_2]$$

- Real solutions

$$\mathbf{x}_1(t) = \operatorname{Re}(\mathbf{z}(t)) = e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2]$$

$$\mathbf{x}_2(t) = \operatorname{Im}(\mathbf{z}(t)) = e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

- General solution

$$\begin{aligned} \mathbf{x}(t) = & C_1 e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ & + C_2 e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2] \end{aligned}$$

Center

- $\alpha = \operatorname{Re}(\lambda) = 0$
- General real solution

$$\begin{aligned}\mathbf{x}(t) = & C_1[\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ & + C_2[\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]\end{aligned}$$

- Every solution is periodic with period $T = 2\pi/\beta$.
- All solution curves are ellipses.
- Direction of flow can be found by looking at the vector field at one point. $(1, 0)^T$ is a good choice.

Spiral Sink

- $\alpha = \operatorname{Re}(\lambda) < 0$
- General real solution

$$\begin{aligned}\mathbf{x}(t) = & C_1 e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ & + C_2 e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]\end{aligned}$$

- All solutions spiral into $\mathbf{0}$ as $t \rightarrow \infty$.

Spiral Source

- $\alpha = \operatorname{Re}(\lambda) > 0$
- General real solution

$$\begin{aligned}\mathbf{x}(t) = & C_1 e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ & + C_2 e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]\end{aligned}$$

- All solutions spiral into $\mathbf{0}$ as $t \rightarrow -\infty$.

Planar Systems

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- Char. polynomial $p(\lambda) = \lambda^2 - T\lambda + D$.
- Eigenvalues

$$\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- λ_1 & λ_2 are the roots of $p(\lambda)$, so

$$\begin{aligned} p(\lambda) &= \lambda^2 - T\lambda + D \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \end{aligned}$$

- $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$.
- Duality between (λ_1, λ_2) and (T, D) .
- Represent systems by location of (T, D) in the TD -plane.

Trace-Determinant Plane

- $T^2 - 4D > 0$
 - ◆ \Rightarrow distinct real eigenvalues λ_1 & λ_2
 - ◆ $D = \lambda_1 \lambda_2 < 0 \Rightarrow$ Saddle point.
 - ◆ $D = \lambda_1 \lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
 - ▶ $T = \lambda_1 + \lambda_2 > 0 \Rightarrow$ Nodal source.
 - ▶ $T = \lambda_1 + \lambda_2 < 0 \Rightarrow$ Nodal sink.

- $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues

$$\lambda = \alpha + i\beta \quad \text{and} \quad \bar{\lambda} = \alpha - i\beta.$$

- ♦ $T = \lambda + \bar{\lambda} = 2\alpha > 0 \Rightarrow$ Spiral source.
- ♦ $T = \lambda + \bar{\lambda} = 2\alpha < 0 \Rightarrow$ Spiral sink.
- ♦ $T = \lambda + \bar{\lambda} = 2\alpha = 0 \Rightarrow$ Center.

Types of Equilibrium Points

- *Generic* types
 - ◆ Saddle, nodal source, nodal sink, spiral source, and spiral sink.
 - ◆ All occupy large open subsets of the trace-determinant plane.
- *Nongeneric* types
 - ◆ Center and many others. Occupy pieces of the boundaries between the generic types.
 - ◆ Project #3 asks you to find the rest of the types.

Higher Dimensional Systems

$$\mathbf{x}' = A\mathbf{x}$$

- A is a real $n \times n$ matrix.
- If λ is an eigenvalue and $\mathbf{v} \neq 0$ is an associated eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Much like the planar case, but now we need n linearly independent solutions.
- We no longer have the easy way to compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.

Proposition: Suppose that $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of A , and that $\mathbf{v}_1, \dots, \mathbf{v}_k$ are associated nonzero eigenvectors. Then $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent.

Theorem: Suppose the $n \times n$ real matrix A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$, and that $\mathbf{v}_1, \dots, \mathbf{v}_n$ are associated nonzero eigenvectors. Then the exponential solutions $\mathbf{x}_i(t) = e^{\lambda_i t} \mathbf{v}_i$, $1 \leq i \leq n$ form a fundamental set of solutions for the system $\mathbf{x}' = A\mathbf{x}$.

Example

$$A = \begin{pmatrix} 17 & -30 & -8 \\ 16 & -29 & -8 \\ -12 & 24 & 7 \end{pmatrix}$$

- Use MATLAB.

Complex Eigenvalues

A real $n \times n$ matrix with a complex eigenvalue λ and associate eigenvector \mathbf{w} .

- $\Rightarrow \bar{\lambda}$ is an eigenvalue and $\bar{\mathbf{w}}$ is an associated nonzero eigenvector.
- Complex valued solutions: $\mathbf{z}(t) = e^{\lambda t} \mathbf{w}$
 $\bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}}$.
- Real solutions: $\mathbf{x}(t) = \text{Re}(\mathbf{z}(t))$
 $\mathbf{y}(t) = \text{Im}(\mathbf{z}(t))$.

Example

$$A = \begin{pmatrix} 21 & 10 & 4 \\ -70 & -31 & -10 \\ 30 & 10 & -1 \end{pmatrix}$$

- The **theorem** applies if some of the eigenvalues are complex and we replace complex conjugate pairs of solutions by their real and imaginary parts.

Repeated Eigenvalues – Example 1

$$A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$
 - ♦ Eigenspace has dimension 1 \Rightarrow one exponential solution

$$\mathbf{x}_1(t) = e^{-3t}(-1/3, 2/3, 1)^T$$

- $\lambda_2 = -1$
 - ♦ Eigenspace has dimension 2 \Rightarrow two linearly independent exponential solutions
 - ♦ Eigenspace has basis $\mathbf{v}_2 = (-5/2, 1, 0)^T$ and $\mathbf{v}_3 = (3/2, 0, 1)^T$.
 - ♦ Linearly independent solutions

$$\mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -5/2 \\ 1 \\ 0 \end{pmatrix} \quad \& \quad \mathbf{x}_3(t) = e^{-t} \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}$$

- \mathbf{x}_1 , \mathbf{x}_2 , and \mathbf{x}_3 are a fundamental set of solutions.

Repeated Eigenvalues – Example 2

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$
 - ♦ Eigenspace has dimension 1 \Rightarrow one exponential solution

$$\mathbf{x}_1(t) = e^{-3t}(-1/2, 3/2, 1)^T$$

- $\lambda_2 = -1$
 - ◆ Eigenspace has dimension 1 \Rightarrow only one exponential solution

$$\mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$$

- Need a third solution.
- Need a new idea.

Multiplicities

A an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \dots, \lambda_k$.
- The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1} (\lambda - \lambda_2)^{q_2} \cdot \dots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of λ_j is q_j .
- The *geometric multiplicity* of λ_j is d_j , the dimension of the eigenspace of λ_j .

- We always have:
 - ◆ $q_1 + q_2 + \cdots + q_k = n.$
 - ◆ $1 \leq d_j \leq q_j.$
 - ◆ There are d_j linearly independent exponential solutions corresponding to $\lambda_j.$
 - ◆ If $d_j = q_j$ for all j we have n linearly independent solutions.
- If $d_j < q_j$ we have trouble.