Math 211

Lecture #30

The Exponential of a Matrix

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Repeated Eigenvalues – Example 1

$$A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$: Eigenspace has dimension 1, with basis $\mathbf{v}_1$, so there is one exponential solution, $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$.
- $\lambda_2 = -1$: Eigenspace has dimension 2 with basis $\mathbf{v}_2$ and $\mathbf{v}_3$, so there are two linearly independent exponential solutions $\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$ and $\mathbf{x}_3(t) = e^{\lambda_2 t}\mathbf{v}_3$.
- $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are a fundamental set of solutions.

Repeated Eigenvalues – Example 2

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$
  - Eigenspace has dimension 1 $\Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{-3t}(-1/2, 3/2, 1)^T$$

- $\lambda_2 = -1$
  - Eigenspace has dimension 1 $\Rightarrow$ only one exponential solution

$$\mathbf{x}_2(t) = e^{-t}\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$$

- Need a third solution.
- Need a new idea.

Multiplicities

A an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of λ_j is q_j.
- The *geometric multiplicity* of λ_j is d_j, the dimension of the eigenspace of λ_j.

- We always have:
  - $q_1 + q_2 + \cdots + q_k = n$.
  - $1 \leq d_j \leq q_j$.
  - There are d_j linearly independent exponential solutions corresponding to λ_j.
  - If $d_j = q_j$ for all j we have n linearly independent solutions.
- If $d_j < q_j$ we have trouble.

New Approach

- $D = 1 : x' = ax$
  - Solution $x(t) = Ce^{at}$.
- $D > 1 : \mathbf{x}' = A\mathbf{x}$
  - Tried $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.
    - Worked well except when eigenvalues have multiplicity greater than 1.
  - Why not $\mathbf{x}(t) = e^{tA}\mathbf{v}$?
- But what is e^{tA}?

Return

Exponential of a Matrix

Definition: The *exponential* of the $n \times n$ matrix A is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$
$$= \sum_0^\infty \frac{1}{n!}A^n.$$

- Example:
  - ♦ $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \Rightarrow e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}.$
  - ♦ $e^{\lambda I} = e^{\lambda} I. \quad e^{0I} = I.$

Properties

- A commutes with e^A,

$$Ae^A = e^A A.$$

- If A and B commute (i.e., $AB = BA$), then

$$e^{A+B} = e^A \cdot e^B.$$

- The inverse of e^A is e^{-A}.
- $\frac{d}{dt} e^{tA} = Ae^{tA}$.

Important Fact

Theorem: The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is $\mathbf{x}(t) = e^{tA}\mathbf{v}$.

- However computing e^{tA} is not easy.

Key to Computing e^{tA} or $e^{tA}\mathbf{v}$

Suppose that A an $n \times n$ matrix, and λ a number (an eigenvalue).

- $A = \lambda I + (A - \lambda I)$; (λI & $A - \lambda I$ commute.)

$$
\begin{aligned}
e^{tA} &= e^{t[\lambda I + (A - \lambda I)]} \\
&= e^{t\lambda I} \cdot e^{t(A-\lambda I)} \\
&= e^{\lambda t} \cdot e^{t(A-\lambda I)} \\
&= e^{\lambda t} \cdot [I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots]
\end{aligned}
$$

$e^{tA}\mathbf{v}$, $\mathbf{v}$ an Eigenvector

Let λ be an eigenvalue and $\mathbf{v}$ an associated eigenvector. Then $(A - \lambda I)\mathbf{v} = \mathbf{0}$, so

$$
\begin{aligned}
e^{tA}\mathbf{v} &= e^{\lambda t} \cdot e^{t(A-\lambda I)}\mathbf{v} \\
&= e^{\lambda t}[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots]\mathbf{v} \\
&= e^{\lambda t}[\mathbf{v} + t(A - \lambda I)\mathbf{v} + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v} + \cdots] \\
&= e^{\lambda t}\mathbf{v}
\end{aligned}
$$

- The infinite series truncates, so we can compute $e^{tA}\mathbf{v}$.

Matrices with One Eigenvalue

A has characteristic polynomial $p(\lambda) = (\lambda - \lambda_1)^n$.

- *Cayley-Hamilton Theorem:* If $p(\lambda)$ is the characteristic polynomial of the matrix A then $p(A) = 0I$.
- In our case $(A - \lambda_1 I)^n = 0I$, so

$$e^{tA} = e^{\lambda_1 t} \cdot [I + t(A - \lambda_1 I) + \frac{t^2}{2!}(A - \lambda_1 I)^2 + \cdots + \frac{t^{n-1}}{(n-1)!}(A - \lambda_1 I)^{n-1}]$$

Example 3

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 2)^2$.

$$A + 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (A + 2I)^2 = 0I$$

$$\begin{aligned} e^{tA} &= e^{-2t}[I + t(A + 2I)] \\ &= e^{-2t} \begin{pmatrix} 1 - t & t \\ -t & 1 + t \end{pmatrix}. \end{aligned}$$

Example 4

$$A = \begin{pmatrix} 0 & -9 & 27 \\ -2 & 3 & -18 \\ -1 & 3 & -12 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)^3. \quad (A + 3I)^2 = 0I.$

$$\begin{aligned} e^{tA} &= e^{-3t}[I + t(A + 3I)] \\ &= e^{-3t} \begin{pmatrix} 1+3t & -9t & 27t \\ -2t & 1+6t & -18t \\ -t & 3t & 1-9t \end{pmatrix}. \end{aligned}$$

Example 2, Reprise

- Distinct eigenvalues $\lambda_1 = -3$ & $\lambda_2 = -1$
- Different from previous two examples.
- $\lambda_1 = -3$ has algebraic multiplicity 1, and geometric multiplicity 1. So there is one exponential solution

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-3t}(-1/2, 3/2, 1)^T.$$

- $\lambda_2 = -1$ has algebraic multiplicity 2, and geometric multiplicity 1. So there is only one exponential solution

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{-t}(-1/2, 1, 1)^T.$$

- However, $\mathrm{null}((A - \lambda_2 I)^2)$ has dimension 2, with basis $(0, 1, 1)^T)$ and $(1, 0, 0)^T$.
  - $\mathbf{v}_2$ is in $\mathrm{null}((A - \lambda_2 I)^2)$
  - Set $\mathbf{v}_3 = (1, 0, 0)^T$.

- If $\mathbf{v} \in \text{null}((A - \lambda_2 I)^2)$ then

$$
\begin{aligned}
e^{tA}\mathbf{v} &= e^{\lambda_2 t}[I + t(A - \lambda_2 I) + \frac{t^2}{2!}(A - \lambda_2 I)^2 + \cdots]\mathbf{v} \\
&= e^{\lambda_2 t}[\mathbf{v} + t(A - \lambda_2 I)\mathbf{v}].
\end{aligned}
$$

- Using $\mathbf{v}_3$ we get the third solution

$$
\begin{aligned}
\mathbf{x}_3(t) &= e^{tA}\mathbf{v}_3 \\
&= e^{-t}[\mathbf{v}_3 + t(A + I)\mathbf{v}_3] \\
&= e^{-t}(1 + 2t, -4t, -4t)^T.
\end{aligned}
$$

- $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are a fundamental set of solutions.

Summary

- In Examples 3 & 4 the matrix has one eigenvalue.
  - The series for $e^{t(A-\lambda I)}$ truncates to a finite sum.
- In Example 2 the matrix had two eigenvalues.
  - The series for $e^{t(A-\lambda I)}$ does not truncate for any λ.
  - However, the series for $e^{t(A-\lambda_2 I)}\mathbf{v}$ does truncate if $(A-\lambda_2 I)^2\mathbf{v} = \mathbf{0}$.

Generalized Eigenvectors

Definition: If λ is an eigenvalue of A and $(A - \lambda I)^p \mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$, then $\mathbf{v}$ is called a *generalized eigenvector* associated with λ.

- The series for $e^{t(A-\lambda I)}\mathbf{v}$ truncates to a finite sum if $\mathbf{v}$ is a generalized eigenvector associated with λ.
- We can compute $e^{tA}\mathbf{v}$.

Return

Theorem: If λ is an eigenvalue of A with algebraic multiplicity q, then there is an integer $p \leq q$ such that $\text{null}((A - \lambda I)^p)$ has dimension q.

- For each generalized eigenvector $\mathbf{v}$ we can compute $e^{tA}\mathbf{v}$.
- We can find q linearly independent solutions associated with the eigenvalue λ.

Procedure for λ of algebraic multiplicity q

To find q linearly independent solutions associated with λ:

- Find the smallest integer p such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_q$ of $\mathrm{null}((A - \lambda I)^p)$.
- For $j = 1, 2, \ldots, q$

$$
\begin{aligned}
\mathbf{x}_j(t) &= e^{tA}\mathbf{v}_j \\
&= e^{\lambda t}[\mathbf{v}_j + t(A - \lambda I)\mathbf{v}_j + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v}_j \\
&\quad + \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{v}_j]
\end{aligned}
$$

Example

- Use MATLAB.

Procedure for a Complex Eigenvalue

If λ is complex of algebraic multiplicity q. Then $\overline{\lambda}$ also has multiplicity q.

- Find the smallest integer p such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q$ of $\mathrm{null}((A - \lambda I)^p)$.
- For $j = 1, 2, \ldots, q$

$$\mathbf{z}_j(t) = e^{tA}\mathbf{w}.$$

Return

- For $j = 1, 2, \ldots, q$

$$\begin{aligned}\mathbf{z}_j(t) = e^{\lambda t}[\mathbf{w}_j &+ t(A - \lambda I)\mathbf{w}_j \\ &+ \frac{t^2}{2!}(A - \lambda I)^2\mathbf{w}_j + \cdots \\ &+ \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{w}_j]\end{aligned}$$

- For $j = 1, 2, \ldots, q$ set

$$\mathbf{x}_j(t) = \mathrm{Re}(\mathbf{z}_j(t)) \quad \text{and}$$

$$\mathbf{y}_j(t) = \mathrm{Im}(\mathbf{z}_j(t)).$$