Math 211

Lecture #31

Stability of Solutions
Higher Order Equations

November 9, 2001

Exponential of a Matrix

Definition: The *exponential* of the $n \times n$ matrix A is the $n \times n$ matrix

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= \sum_0^\infty \frac{1}{n!}A^n. \end{aligned}$$

Theorem: The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is $\mathbf{x}(t) = e^{tA}\mathbf{v}$.

Return

Computing $e^{tA}\mathbf{v}$

- If λ is an eigenvalue and $\mathbf{v}$ is an associated eigenvector, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
- If $(A - \lambda I)^p\mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$, then

$$e^{tA}\mathbf{v} = e^{\lambda t}\Big[\mathbf{v} + +t(A-\lambda I)\mathbf{v} + \frac{t^2}{2!}(A-\lambda I)^2\mathbf{v} \\ + \cdots + \frac{t^{p-1}}{(p-1)!}(A-\lambda I)^{p-1}\mathbf{v}\Big]$$

Return

Generalized Eigenvectors

Definition: If λ is an eigenvalue of A and $(A-\lambda I)^p\mathbf{v}=\mathbf{0}$ for some integer $p \geq 1$, then $\mathbf{v}$ is called a *generalized eigenvector* associated with λ.

- We can compute $e^{tA}\mathbf{v}$ for all such $\mathbf{v}$.

Theorem: If λ is an eigenvalue of A with algebraic multiplicity q, then there is an integer $p \leq q$ such that $\mathrm{null}((A-\lambda I)^p)$ has dimension q.

- We can find q linearly independent solutions associated with the eigenvalue λ.

Procedure for λ of algebraic multiplicity q

To find q linearly independent solutions associated with λ:

- Find the smallest integer p such that $\operatorname{null}((A - \lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_q$ of $\operatorname{null}((A - \lambda I)^p)$.
- For $j = 1, 2, \ldots, q$

$$\begin{aligned} \mathbf{x}_j(t) &= e^{tA}\mathbf{v}_j \\ &= e^{\lambda t}\Big[\mathbf{v}_j + t(A - \lambda I)\mathbf{v}_j + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v}_j \\ &\qquad + \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{v}_j\Big] \end{aligned}$$

Example

- Use MATLAB.

Procedure for a Complex Eigenvalue

If λ is a complex eigenvalue of algebraic multiplicity q. Then $\overline{\lambda}$ also has algebraic multiplicity q.

- Find the smallest integer p such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q$ of $\mathrm{null}((A - \lambda I)^p)$.

- For $j = 1, 2, \ldots, q$ we have solutions

$$\begin{aligned}
\mathbf{z}_j(t) &= e^{tA}\mathbf{w}_j \\
&= e^{\lambda t}\Big[\mathbf{w}_j + t(A-\lambda I)\mathbf{w}_j + \frac{t^2}{2!}(A-\lambda I)^2\mathbf{w}_j \\
&\quad + \cdots + \frac{t^{p-1}}{(p-1)!}(A-\lambda I)^{p-1}\mathbf{w}_j\Big]
\end{aligned}$$

- $\mathbf{z}_1, \ldots, \mathbf{z}_q$ together with $\overline{\mathbf{z}_1}, \ldots, \overline{\mathbf{z}_q}$ are $2q$ linearly independent complex valued solutions.
- For $j = 1, 2, \ldots, q$ set $\mathbf{x}_j(t) = \mathrm{Re}(\mathbf{z}_j(t))$ and $\mathbf{y}_j(t) = \mathrm{Im}(\mathbf{z}_j(t))$. These are $2q$ linearly independent real valued solutions.

Stability

Autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with an equilibrium point at $\mathbf{x}_0$.

- Basic question: What happens to *all solutions* as $t \to \infty$?
- $\mathbf{x}_0$ is *stable* if for every $\epsilon > 0$ there is a $\delta > 0$ such that a solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0| < \delta \ \Rightarrow \ |\mathbf{x}(t) - \mathbf{x}_0| < \epsilon$ for all $t \geq 0$.
    - Every solution that starts close to $\mathbf{x}_0$ stays close to $\mathbf{x}_0$.

Return

- $\mathbf{x}_0$ is *asymptotically stable* if it is stable and there is an $\eta > 0$ such that if $\mathbf{x}(t)$ is a solution with $|\mathbf{x}(0) - \mathbf{x}_0| < \eta$, then $\mathbf{x}(t) \to \mathbf{x}_0$ as $t \to \infty$.
  - $\mathbf{x}_0$ is called a *sink.*
  - Every solution that starts close to $\mathbf{x}_0$ approaches $\mathbf{x}_0$.
- $\mathbf{x}_0$ is *unstable* if there is an $\epsilon > 0$ such that for any $\delta > 0$ there is a solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$ with the property that there are values of $t > 0$ such that $|\mathbf{x}(t) - \mathbf{x}_0| > \epsilon$.
  - There are solutions starting arbitrarily close to $\mathbf{x}_0$ that move away from $\mathbf{x}_0$.

Return

Examples $D = 2$

- Sinks are asymptotically stable.
  - The eigenvalues have negative real part.
- Sources are unstable.
  - The eigenvalues have positive real part.
- Saddles are unstable.
  - One eigenvalue has positive real part.
- Centers are stable but not asymptotically stable.
  - The eigienvalues have real part $= 0$.

Return

Theorem: Let A be an $n \times n$ real matrix.

- Suppose the real part of every eigenvalue of A is negative. Then $\mathbf{0}$ is an asymptotically stable equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.
- Suppose A has at least one eigenvalue with positive real part. Then $\mathbf{0}$ is an unstable equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.

Examples

- $D = 2$
  - $T^2 - 4D = 0.$
    - $T < 0 \Rightarrow$ sink. $T > 0 \Rightarrow$ source.
- $\mathbf{y}' = A\mathbf{y}$,

$$A = \begin{pmatrix} -2 & -18 & -7 & -14 \\ 1 & 6 & 2 & 5 \\ 2 & 2 & -3 & 0 \\ -2 & -8 & -1 & -6 \end{pmatrix}.$$

  - A has eigenvalues -1, -2, & $-1 \pm i$.
  - $\mathbf{0}$ is asymptotically stable.

Higher Order Equations

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

- Second order: $y'' + py' + qy = 0.$
- Equivalent system: $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

  ◆ A fundamental set of solutions for the system consists of two linearly independent solutions.

Return

Linear Independence

Definition: Two functions $u(t)$ and $v(t)$ are *linearly independent* if neither is a constant multiple of the other.

- $u(t)$ and $v(t)$ are linearly independent solutions to $y'' + py' + qy = 0 \Leftrightarrow \begin{pmatrix} u \\ u' \end{pmatrix}$ & $\begin{pmatrix} v \\ v' \end{pmatrix}$ are linearly independent solutions to the equivalent system.

Return

General Solution

Theorem: Suppose that $y_1(t)$ & $y_2(t)$ are linearly independent solutions to the equation

$$y'' + py' + qy = 0.$$

Then the general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

Definition: A set of two linearly independent solutions is called a *fundamental set of solutions.*

Return

Solutions to $y'' + py' + qy = 0$.

- Equivalent system: $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

- Look for exponential solutions $y(t) = e^{\lambda t}$.
- *Characteristic equation:* $\lambda^2 + p\lambda + q = 0$.
- *Characteristic polynomial:* $\lambda^2 + p\lambda + q$.
- Same for the 2^{nd} order equation and the system.

Return

Real Roots

- If λ is a root to the characteristic polynomial then $y(t) = e^{\lambda t}$ is a solution.
- If λ is a root to the characteristic polynomial of multiplicity 2, then $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ are linearly independent solutions.

Complex Roots

- If $\lambda = \alpha + i\beta$ is a complex root of the characteristic equation , then so is $\overline{\lambda} = \alpha - i\beta$.
- A complex valued fundamental set of solutions is

$$z(t) = e^{\lambda t} \quad \text{and} \quad \overline{z}(t) = e^{\overline{\lambda} t}.$$

- A real valued fundamental set of solutions is

$$x(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y(t) = e^{\alpha t} \sin \beta t.$$

Return

General solution

Examples

- $y'' - 5y' + 6y = 0.$
- $y'' + 25y = 0.$
- $y'' + 4y' + 13y = 0.$

Key to Computing e^{tA} or $e^{tA}\mathbf{v}$

Suppose that A an $n \times n$ matrix, and λ a number (an eigenvalue).

- $A = \lambda I + (A - \lambda I)$; ($\lambda I$ & $A - \lambda I$ commute.)

$$
\begin{aligned}
e^{tA} &= e^{t[\lambda I + (A - \lambda I)]} \\
&= e^{t\lambda I} \cdot e^{t(A - \lambda I)} \\
&= e^{\lambda t} \cdot e^{t(A - \lambda I)} \\
&= e^{\lambda t} \cdot \left[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots\right]
\end{aligned}
$$

Return

Multiplicities

A an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of λ_j is q_j.
- The *geometric multiplicity* of λ_j is d_j, the dimension of the eigenspace of λ_j.

Return