

# Math 211

Lecture #33

Harmonic Motion  
Inhomogeneous Equations

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## Harmonic Motion

- Spring:  $y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t)$ .
- Circuit:  $I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t)$ .
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2x = f(t).$$

- ◆ We call this the equation for *harmonic motion*.
- ◆ It includes both the vibrating spring and the *RLC* circuit.

# The Equation for Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

- $\omega_0$  is the *natural frequency*.
  - ♦ Spring:  $\omega_0 = \sqrt{k/m}$ .
  - ♦ Circuit:  $\omega_0 = \sqrt{1/LC}$ .
- $c$  is the *damping constant*.
  - ♦ Spring:  $2c = \mu/m$ .
  - ♦ Circuit:  $2c = R/L$ .
- $f(t)$  is the *forcing term*.

# Simple Harmonic Motion

No **forcing**, and no damping.

$$x'' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + \omega_0^2$ ,  $\lambda = \pm i\omega_0$ .
- Fundamental set of solutions:  $x_1(t) = \cos \omega_0 t$  &  $x_2(t) = \sin \omega_0 t$ .
- General solution:  $x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$ .
- Every solution is periodic at the natural **frequency**  $\omega_0$ .
  - ♦ The period is  $T = 2\pi/\omega_0$ .

## Amplitude and Phase

- Put  $C_1$  and  $C_2$  in polar coordinates:

$$C_1 = A \cos \phi, \text{ \& } C_2 = A \sin \phi.$$

- Then  $x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$

$$= A \cos(\omega_0 t - \phi).$$

- $A$  is the *amplitude*;  $A = \sqrt{C_1^2 + C_2^2}$ .
- $\phi$  is the *phase*;  $\tan \phi = C_2/C_1$ .

## Examples

- $C_1 = 3, C_2 = 4 \Rightarrow A = 5, \phi = 0.9273.$
- $C_1 = -3, C_2 = 4 \Rightarrow A = 5, \phi = 2.2143.$
- $C_1 = -3, C_2 = -4 \Rightarrow A = 5, \phi = -2.2143.$

## Example

$$x'' + 16x = 0, x(0) = -2 \text{ \& } x'(0) = 4$$

- Natural frequency:  $\omega_0^2 = 16 \Rightarrow \omega_0 = 4$ .
- General solution:  $x(t) = C_1 \cos 4t + C_2 \sin 4t$ .
- IC:  $-2 = x(0) = C_1$ , and  $4 = x'(0) = 4C_2$ .
- Solution

$$\begin{aligned} x(t) &= -2 \cos 2t + \sin 2t \\ &= \sqrt{5} \cos(2t - 2.6779). \end{aligned}$$

# Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$ ; roots  $-c \pm \sqrt{c^2 - \omega_0^2}$ .
- Three cases
  - ♦  $c < \omega_0$  *Underdamped*
  - ♦  $c > \omega_0$  *Overdamped*
  - ♦  $c = \omega_0$  *Critically damped*

## Underdamped

- $c < \omega_0$
- Two complex roots  $\lambda$  and  $\bar{\lambda}$ , where  $\lambda = -c + i\omega$  and  $\omega = \sqrt{\omega_0^2 - c^2}$ .
- General solution

$$\begin{aligned}x(t) &= e^{-ct}[C_1 \cos \omega t + C_2 \sin \omega t] \\ &= Ae^{-ct} \cos(\omega t - \phi)\end{aligned}$$

# Overdamped

- $c > \omega_0$ , so two real roots

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2}$$

$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

- $\lambda_1 < \lambda_2 < 0$ .
- General solution

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

# Critically Damped

- $c = \omega_0$
- One negative real root  $\lambda = -c$  with multiplicity 2.
- General solution

$$x(t) = e^{-ct}[C_1 + C_2t].$$

# Inhomogeneous Equations

$$y'' + py' + qy = f(t)$$

- Corresponding homogeneous equation

$$y'' + py' + qy = 0$$

- ◆ We know how to find a fundamental set of solutions  $y_1$  and  $y_2$ .
- ◆ The general solution of the homogeneous equation is  $y_h(t) = C_1y_1(t) + C_2y_2(t)$ .

**Theorem:** Assume

- $y_p(t)$  is a particular solution to the inhomogeneous equation  $y'' + py' + qy = f(t)$ ;
- $y_1(t)$  &  $y_2(t)$  is a fundamental set of solutions to the homogeneous equation  $y'' + py' + qy = 0$ .

Then the general solution to the inhomogeneous equation is

$$y(t) = y_p(t) + C_1y_1(t) + C_2y_2(t).$$

## Method of Undetermined Coefficients

$$y'' + py' + qy = f(t)$$

- If the forcing term  $f(t)$  has a form which is replicated under differentiation, then look for a particular solution of the same general form as the forcing term.

# Exponential Forcing Term

$$y'' + py' + qy = Ce^{at}$$

- Example:  $y'' + 3y' + 2y = 4e^{-3t}$
- Try  $y_p(t) = ae^{-3t}$ ;  $a$  to be determined.
  - ◆ Particular solution:  $y_p(t) = 2e^{-3t}$ .
- Homogeneous equation:  $y'' + 3y' + 2y = 0$ .
  - ◆ Fundamental set of solutions:  $e^{-2t}$  &  $e^{-t}$ .
- General solution to the inhomogeneous equation:

$$y(t) = 2e^{-3t} + C_1e^{-t} + C_2e^{-2t}.$$

# Trigonometric Forcing Term

$$y'' + py' + qy = A \cos \omega t + B \sin \omega t$$

- Example:  $y'' + 4y' + 5y = 4 \cos 2t - 3 \sin 2t$
- Try  $y_p(t) = a \cos 2t + b \sin 2t$ 
  - ◆ Particular solution:  $y_p(t) = [28 \cos 2t + 29 \sin 2t]/65$ .
- Homogeneous equation:  $y'' + 4y' + 5y = 0$ 
  - ◆ Fund. set of sol'ns:  $e^{-2t} \cos t$  &  $e^{-2t} \sin t$ .
- General solution to the inhomogeneous equation:

$$y(t) = \frac{28 \cos 2t + 29 \sin 2t}{65} + e^{-2t} [C_1 \cos t + C_2 \sin t].$$

## Complex Method

$$x'' + px' + qx = A \cos \omega t \quad \text{or}$$

$$y'' + py' + qy = A \sin \omega t.$$

- Solve  $z'' + pz' + qz = Ae^{i\omega t}$ .
  - ◆ Try  $z(t) = ae^{i\omega t}$ .
- $x_p(t) = \operatorname{Re}(z(t))$  and  $y_p(t) = \operatorname{Im}(z(t))$ .

## Example

$$x'' + 4x' + 5x = 4 \cos 2t$$

- Solve  $z'' + 4z' + 5z = 4e^{2it}$ .
  - ◆ Try  $z(t) = ae^{2it}$ .
  - ◆ Particular solution:  $z(t) = (4 - 32i)e^{2it}/65$ .
- Particular solution to the real equation:

$$\begin{aligned}x_p(t) &= \operatorname{Re}(z(t)) \\ &= [4 \cos 2t + 32 \sin 2t] / 65.\end{aligned}$$

## Polynomial Forcing Term

$$y'' + py' + qy = P(t)$$

- Example:  $y'' - 3y' + 2y = 1 - 4t$ .
  - ◆ Try  $y(t) = a + bt$ .
  - ◆ Particular solution:  $y(t) = -5 - 2t$ .
- General solution

$$y(t) = -5 - 2t + C_1e^t + C_2e^{2t}.$$

## Exceptional Cases

- Example:  $y'' - 3y' + 2y = 3e^t$ .
  - ◆ Try  $y(t) = ae^t$
  - ◆ The method does not work because  $e^t$  is a solution to the associated homogeneous equation.
- Try  $y(t) = ate^t$ 
  - ◆ Particular solution:  $y_p(t) = -3te^t$ .
- General solution:  $y(t) = -3te^t + C_1e^t + C_2e^{2t}$ .
- If the suggested particular solution does not work, multiply it by  $t$  and try again.

## Combination Forcing Term

Example  $y'' + 5y' + 6y = 2e^{2t} - 5 \cos t$

- Solve

$$y_1'' + 5y_1' + 6y_1 = 2e^{2t}$$

$$y_2'' + 5y_2' + 6y_2 = -5 \cos t$$

- Set  $y(t) = y_1(t) + y_2(t)$ .