

# Math 211

Lecture #35

Forced Harmonic Motion

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# Forced Harmonic Motion

Assume an oscillatory forcing term:

$$y'' + 2cy' + \omega_0^2 y = A \cos \omega t$$

- $A$  is the forcing amplitude
- $\omega$  is the forcing frequency
- $\omega_0$  is the natural frequency.
- $c$  is the damping constant.

# Forced Undamped Motion

$$y'' + \omega_0^2 y = A \cos \omega t$$

- Homogeneous equation:  $y'' + \omega_0^2 y = 0$ 
  - ◆ General solution

$$y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

- $\omega \neq \omega_0$ : Particular solution

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

- $\omega \neq \omega_0$

- ◆ Initial conditions  $x(0) = x'(0) = 0 \Rightarrow$

$$x(t) = \frac{A}{\omega_0^2 - \omega^2} [\cos \omega t - \cos \omega_0 t].$$

- ◆ Set  $\bar{\omega} = \frac{\omega_0 + \omega}{2}$  and  $\delta = \frac{\omega_0 - \omega}{2}$ .

$$x(t) = \frac{A \sin \delta t}{2\bar{\omega}\delta} \sin \bar{\omega} t.$$

- ◆ Fast oscillation with frequency  $\bar{\omega}$  with amplitude oscillating slowly with frequency  $\delta$ .
  - ▶ Beats.

- $\omega = \omega_0$

$$y'' + \omega_0^2 y = A \cos \omega_0 t.$$

- ◆ An exceptional case. Particular solution

$$x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t.$$

- ◆ Oscillation with increasing amplitude.
- ◆ First example of *resonance*.
  - ▶ Forcing at the natural frequency can cause oscillations that grow out of control.

## Forced, Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t$$

- Homo. equation:  $x'' + 2cx' + \omega_0^2 x = 0$
- Ch. polynomial:  $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$
- Assume the underdamped case, where  $c < \omega_0$ .
- Roots  $\lambda = -c \pm \sqrt{c^2 - \omega_0^2} = -c \pm i\eta$  where  $\eta = \sqrt{\omega_0^2 - c^2}$ .
- Fundamental set of solutions  $x_1(t) = e^{-ct} \cos \eta t$  and  $x_2(t) = e^{-ct} \sin \eta t$

## Inhomogeneous equation

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t$$

- Use the complex method. Solve

$$z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}.$$

- ♦ Try  $z(t) = ae^{i\omega t}$ .

$$\begin{aligned} z'' + 2cz' + \omega_0^2 z &= [(i\omega)^2 + 2c(i\omega) + \omega_0^2]ae^{i\omega t} \\ &= P(i\omega)z \end{aligned}$$

- ♦  $P(i\omega) = (i\omega)^2 + 2c(i\omega) + \omega_0^2 = [\omega_0^2 - \omega^2] + 2ic\omega$ .

- Complex solution:  $z(t) = \frac{1}{P(i\omega)} Ae^{i\omega t}$ .
- Real solution:  $x_p(t) = \operatorname{Re}(z(t))$ .

## Example

$$x'' + 5x' + 4x = 50 \cos 3t$$

- $P(\lambda) = \lambda^2 + 5\lambda + 4.$ 
  - ♦  $P(i\omega) = P(3i) = -5 + 15i$
- $z(t) = \frac{1}{P(i\omega)} \cdot 50 \cos 3t$ 
$$= -[(\cos 3t - 3 \sin 3t) + i(\sin 3t + 3 \cos 3t)]$$
- $x_p(t) = \operatorname{Re}(z(t))$ 
$$= 3 \sin 3t - \cos 3t.$$

# Transfer Function

- Complex solution:

$$z(t) = \frac{1}{P(i\omega)} A e^{i\omega t} = H(i\omega) A e^{i\omega t}.$$

- $H(i\omega) = \frac{1}{P(i\omega)}$  is called the *transfer function*.
  - ◆ We will write  $H(i\omega) = G(\omega) e^{-i\phi(\omega)}$ .
    - ▶  $G(\omega) = |H(i\omega)|$  is the *gain* and  $\phi$  is the *phase*.

- The characteristic polynomial  $P(i\omega) = Re^{i\phi}$

- ♦  $R = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}$

- ♦  $\phi = \operatorname{arccot} \left( \frac{\omega_0^2 - \omega^2}{2c\omega} \right)$ .

- Transfer Function

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R} e^{-i\phi} = G(\omega) e^{-i\phi}.$$

- ♦ The gain  $G(\omega) = \frac{1}{R} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}$ .

- ♦ The phase shift  $\phi = \operatorname{arccot} \left( \frac{\omega_0^2 - \omega^2}{2c\omega} \right)$ .

- The complex particular solution is

$$\begin{aligned}z(t) &= H(i\omega) A e^{i\omega t} \\ &= G(\omega) e^{-i\phi} \cdot A e^{i\omega t} \\ &= G(\omega) A e^{i(\omega t - \phi)}.\end{aligned}$$

- The real particular solution is

$$\begin{aligned}x_p(t) &= \operatorname{Re}(z(t)) \\ &= G(\omega) A \cos(\omega t - \phi).\end{aligned}$$

- General Solution

$$\begin{aligned}x(t) &= x_p(t) + x_h(t) \\ &= G(\omega)A \cos(\omega t - \phi) \\ &\quad + e^{-ct}[C_1 \cos \eta t + C_2 \sin \eta t].\end{aligned}$$

- *Transient term.*

- ◆  $x_h(t) = e^{-ct}[C_1 \cos \eta t + C_2 \sin \eta t].$

- *Steady-state solution.*

- ◆  $x_p(t) = G(\omega)A \cos(\omega t - \phi).$

- **Example:**  $x'' + 5x' + 4x = A \cos \omega t$

- $G(\omega) = \frac{1}{\sqrt{(4 - \omega^2)^2 + 25\omega^2}}$  and

$$\phi = \operatorname{arccot} \left( \frac{4 - \omega^2}{5\omega} \right).$$

- ◆ With  $\omega = 3$ ,

$$G(3) = \frac{1}{5\sqrt{10}} \approx 0.0632$$

$$\phi = \operatorname{arccot}(-3/5) \approx 2.1112.$$

- ◆ **SS solution**  $x_p(t) = G(3)A \cos(3t - \phi)$ .

# Steady-State Solution

$$x_p(t) = G(\omega)A \cos(\omega t - \phi).$$

- The **forcing function** is  $A \cos \omega t$ .
- The steady-state response is oscillatory.
  - ◆ The amplitude is  $G(\omega)$  times the amplitude of the forcing term.
  - ◆ At the driving frequency.
  - ◆ With a phase shift of  $\phi/\omega$ .

## Gain

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}$$

- Set

$$\omega = s\omega_0 \quad \text{or} \quad s = \frac{\omega}{\omega_0}$$

$$c = \frac{D\omega_0}{2} \quad \text{or} \quad D = \frac{2c}{\omega_0}$$

Then

$$G(\omega) = \frac{1}{\omega_0^2} \frac{1}{\sqrt{(1 - s^2)^2 + D^2 s^2}}$$