Math 211

Lecture #36

Forced Harmonic Motion

Nonlinear Systems

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Forced, Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

- Ch. polynomial: $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$
- General Solution

$$x(t) = G(\omega)A\cos(\omega t - \phi) + x_h(t).$$

  - *Transient term* $x_h(t)$ dies out exponentially.
  - *Steady-state solution* $x_p(t) = G(\omega)A\cos(\omega t - \phi)$.
    - Gain: $G(\omega) = 1/\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}$.
    - Phase: $\phi = \operatorname{arccot}\left((\omega_0^2 - \omega^2)/2c\omega\right)$.

Steady-State Solution

$$x_p(t) = G(\omega)A\cos(\omega t - \phi).$$

- The forcing function is $A\cos\omega t$.
- The steady-state response is oscillatory.
  - The amplitude is $G(\omega)$ times the amplitude of the forcing term.
  - The steady-state oscillation is at the forcing frequency.
  - There is a phase shift of ϕ/ω.

Interacting Species

- Two species with populations x_1 & x_2.
- Interaction between the species can be helpful or detrimental.
- Basic model

$$x_1' = r_1 x_1$$

$$x_2' = r_2 x_2$$

- r_1 & r_2 are the *reproductive rates.*

Reproductive Rates

- If $x_2 = 0$ the reproductive rate for x_1 is

$$r_1 = a_1 - b_1 x_1.$$

  - $a_1 > 0 \Rightarrow$ natural growth.
  - $a_1 < 0 \Rightarrow$ natural decline.
  - $b_1 = 0$ Malthusian growth.
  - $b_1 > 0$ logistic growth.

Return

- If $x_2 > 0$ the reproductive rate for x_1 is

$$r_1 = a_1 - b_1 x_1 + c_1 x_2.$$

  - $c_1 > 0 \Rightarrow$ interaction is helpful to x_1.
  - $c_1 < 0 \Rightarrow$ interaction is detrimental to x_1.
  - The reproductive rate for x_2 is

$$r_2 = a_2 - b_2 x_2 + c_2 x_1.$$

- The model for interacting species is

$$x_1' = (a_1 - b_1 x_1 + c_1 x_2) x_1$$
$$x_2' = (a_2 - b_2 x_2 + c_2 x_1) x_2$$

Predator Prey Model

Rabbits & foxes, fish & sharks, and cottony cushion scale insect & ladybird beetle.

- F = fish & S = sharks.

$$F' = (a - bS)F$$

$$S' = (-c + dF)S$$

or

$$F' = (a - eF - bS)F$$

$$S' = (-c + dF)S$$

$$a = 3,\ b = 3,\ c = 1,\ d = 3,\ e = 3.$$

Competing Species

Cattle and sheep.

- x_1 and x_2 competing for resources.

$$x_1' = (a_1 - b_1 x_1 + c_1 x_2)x_1$$

$$x_2' = (a_2 - b_2 x_2 + c_2 x_1)x_2$$

  - $a_i > 0$, $b_i > 0$, & $c_i < 0$

- Example:

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

Linearization

The principal idea of differential calculus:

- Approximate nonlinear mathematical objects by linear ones.
- Example: Approximate the function $f(y)$ near y_0 by a linear function.

$$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$

$$\text{where } \lim_{h \to 0} \frac{R(h)}{h} = 0.$$

♦ The linear function is $L(h) = f(y_0) + f'(y_0)h.$

Linearization of an ODE

$$y' = f(y)$$

- Assume $f(y_0) = 0$ and $f'(y_0) \neq 0$.
- Set $y = y_0 + u$. Get

$$\begin{aligned} u' &= f(y_0 + u) \\ &= f'(y_0)u + R(u) \end{aligned}$$

- Approximate by the linear differential equation

$$\tilde{u}' = f'(y_0)\tilde{u}$$

- If $f'(y_0) \neq 0$ the equilibrium point of the linearization at 0 has the same stability properties as that of the nonlinear equation at y_0.
  - $f'(y_0) > 0 \;\Rightarrow\; y_0$ is unstable.
  - $f'(y_0) < 0 \;\Rightarrow\; y_0$ is asymptotically stable.
- We can solve the linearization explicitly.

Linearization of a Planar System

$$x' = f(x, y)$$
$$y' = g(x, y)$$

- Asume (x_0, y_0) is an equilibrium point, so

$$f(x_0, y_0) = g(x_0, y_0) = 0$$

We have by Taylor's theorem

$$f(x_0+u, y_0+v) = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$

$$g(x_0+u, y_0+v) = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

where $\dfrac{R_f(u,v)}{\sqrt{u^2+v^2}} \to 0$ and $\dfrac{R_g(u,v)}{\sqrt{u^2+v^2}} \to 0$

- Set $x = x_0 + u$ and $y = y_0 + v$. The system becomes

$$u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$
$$v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

Linearization at (x_0, y_0)

$$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0)\tilde{v}$$
$$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0)\tilde{v}$$

- This is a linear system.
  - We can solve it explicitly.
  - Does it give information about the original nonlinear system?

Matrix Form of the Linearization

Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0, y_0) & \dfrac{\partial f}{\partial y}(x_0, y_0) \\ \dfrac{\partial g}{\partial x}(x_0, y_0) & \dfrac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

Theorem: Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
  - Center and others. Occupy pieces of the boundaries.

Examples

- Predator prey
- Competing species
- Center

$$x' = y + \alpha x(x^2 + y^2)$$
$$y' = -x + \alpha y(x^2 + y^2)$$