Math 211

Lecture #37

Linearization

November 26, 2001

Linearization

The principal idea of differential calculus:

- Approximate nonlinear mathematical objects by linear ones.
- Example: Approximate the function $f(y)$ near y_0 by a linear function.

$$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$

$$\text{where } \lim_{h \to 0} \frac{R(h)}{h} = 0.$$

- ♦ The linear function is $L(h) = f(y_0) + f'(y_0)h$.

Linearization of an ODE

$$y' = f(y)$$

- Assume $f(y_0) = 0$ and $f'(y_0) \neq 0$.
- Set $y = y_0 + u$. Get

$$\begin{aligned} u' &= f(y_0 + u) \\ &= f'(y_0)u + R(u) \end{aligned}$$

- Approximate by the linear differential equation

$$\tilde{u}' = f'(y_0)\tilde{u}$$

- If $f'(y_0) \neq 0$ the equilibrium point of the linearization at 0 has the same stability properties as that of the nonlinear equation at y_0.
  - $f'(y_0) > 0 \Rightarrow y_0$ is unstable.
  - $f'(y_0) < 0 \Rightarrow y_0$ is asymptotically stable.
- We can solve the linearization explicitly.

Linearization of a Planar System

$$x' = f(x, y)$$
$$y' = g(x, y)$$

- (x_0, y_0) is an equilibrium point so

$$f(x_0, y_0) = g(x_0, y_0) = 0$$

We have by Taylor's theorem

$$f(x_0 + u, y_0 + v)$$
$$= \frac{\partial f}{\partial x}(x_0, y_0) \cdot u + \frac{\partial f}{\partial y}(x_0, y_0) \cdot v + R_f(u, v)$$
$$g(x_0 + u, y_0 + v)$$
$$= \frac{\partial g}{\partial x}(x_0, y_0) \cdot u + \frac{\partial g}{\partial y}(x_0, y_0) \cdot v + R_g(u, v)$$

where $\frac{R_f(u, v)}{\sqrt{u^2 + v^2}} \to 0$ and $\frac{R_g(u, v)}{\sqrt{u^2 + v^2}} \to 0$

- Set $x = x_0 + u$ and $y = y_0 + v$. The system becomes

$$u' = \frac{\partial f}{\partial x}(x_0, y_0) \cdot u + \frac{\partial f}{\partial y}(x_0, y_0) \cdot v + R_f(u, v)$$
$$v' = \frac{\partial g}{\partial x}(x_0, y_0) \cdot u + \frac{\partial g}{\partial y}(x_0, y_0) \cdot v + R_g(u, v)$$

Linearization at (x_0, y_0)

$$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \tilde{v}$$
$$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0) \cdot \tilde{v}$$

- This is a linear system.
  - It is called the *linearization of the system at the equilibibirum point* (x_0, y_0)
  - We can solve it explicitly.
  - Does it give information about the original system?

Matrix Form of the Linearization

Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

- The behavior of solutions to the linearization is determined by the eigenvalues of the Jacobian.

Theorem: Consider the planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
  - Center and others, which occupy pieces of the boundaries between the generic points.

Examples

- Predator prey

$$F' = (3 - 3S)F$$
$$S' = (-1 + 3F)S$$

or

$$F' = (3 - 3F - 3S)F$$
$$S' = (-1 + 3F)S$$

- Competing species

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- Center

$$x' = y + \alpha x(x^2 + y^2)$$
$$y' = -x + \alpha y(x^2 + y^2)$$

  - $\alpha > 0 \;\Rightarrow\; (0,0)^T$ is unstable.
  - $\alpha < 0 \;\Rightarrow\; (0,0)^T$ is a sink.