Math 211

Lecture #38

Linearization in Higher Dimension

November 28, 2001

Linearization of a Planar System

$$x' = f(x, y)$$
$$y' = g(x, y)$$

- (x_0, y_0) is an eq. point so $f(x_0, y_0) = g(x_0, y_0) = 0$.
- Linearization at (x_0, y_0)

$$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \tilde{v}$$
$$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0) \cdot \tilde{v}$$

Matrix Form of the Linearization

Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

Theorem: Consider the planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - Occupy large open subsets of the trace-determinant plane.
- Nongeneric types: center and others.
  - Occupy pieces of the boundaries between the generic points.
  - Example:

$$x' = y + \alpha x(x^2 + y^2)$$
$$y' = -x + \alpha y(x^2 + y^2)$$

Higher Dimensional Systems

Autonomous equation $\mathbf{y}' = \mathbf{f}(\mathbf{y})$

- $\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$
- $\mathbf{f}(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \cdots, f_n(\mathbf{y}))^T$
- J is the Jacobian matrix

$$\mathbf{f}(\mathbf{y}_0 + \mathbf{u}) = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u}) \quad \text{where} \quad \lim_{\mathbf{u}\to\mathbf{0}} \frac{\mathbf{R}(\mathbf{u})}{|\mathbf{u}|} = \mathbf{0}.$$

- Set $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$. The system becomes

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u}).$$

- The linearization is $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$.

Theorem: Suppose that $\mathbf{y}_0$ is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of $\mathbf{f}$ at $\mathbf{y}_0$.

1. Suppose that the real part of every eigenvalue of J is negative. Then $\mathbf{y}_0$ is an asymptotically stable equilibrium point.
2. Suppose that J has at least one eigenvalue with positive real part. Then $\mathbf{y}_0$ is an unstable equilibrium point.

Example

$$x' = -2x - 4y + 2xy$$
$$y' = x - 6y + x^2 - y^2$$

- The origin $(0, 0)$ is an equilibrium point.
- The Jacobian has one eigenvalue, $\lambda = -4$, of algebraic multiplicity 2.
- First theorem does not apply.
- Second theorem $\Rightarrow$ the origin is a sink.

The Lorenz System

$$x' = -ax + ay$$
$$y' = rx - y - xz$$
$$z' = -bz + xy$$

- Equilibrium points.
  - $(r \leq 1)$ $(0, 0, 0)$
  - $(r > 1)$ Set $s = \sqrt{b(r-1)}$. The equilibrium points are $(0, 0, 0)$, and $\mathbf{c}^{\pm} = (\pm s, \pm s, r - 1)$.

- The Jacobian is

$$J = \begin{pmatrix} -a & a & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

  - Use $a = 10$ and $b = 8/3$.
  - $(0, 0, 0)$
    - If $r < 1$ $(0, 0, 0)$ is asymptotically stable.
    - If $r > 1$ $(0, 0, 0)$ is unstable.

  - $\mathbf{c}^+$ and $\mathbf{c}^-$
    - For $1 < r < 470/19 \approx 24.74$, $\mathbf{c}^+$ and $\mathbf{c}^-$ are asymptotically stable.
    - For $r > 470/19 \approx 24.74$, $\mathbf{c}^+$ and $\mathbf{c}^-$ are unstable.

  - As r varies the Lorenz system displays a wide variety of behaviors.
    - For $r = 28$ we have Lorenz's strange attractor.
    - For $r = 100$ there is a periodic attractor.
    - For $r = 200$ there is another strange attractor.

Invariant Sets

Definition: A set S is *(positively) invariant* for the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{y}(0) = \mathbf{y}_0 \in S$ implies that $\mathbf{y}(t) \in S$ for all $t \geq 0$.

- Examples:
  - An equilibrium point.
  - Any solution curve.

Return

Example — Competing Species

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- The positive x- and y-axes are invariant.
- The positive quadrant is invariant.
  - Populations should remain nonnegative.
- The set $S = \{(x, y) \,|\, 0 < x < 3,\ 0 < y < 3\}$ is positively invariant.

Return

Nullclines

$$x' = f(x, y)$$
$$y' = g(x, y)$$

Definition: The x-*nullcline* is the set defined by $f(x, y) = 0$. The y-*nullcline* is the set defined by $g(x, y) = 0$.

- Along the x-nullcline the vector field points up or down.
- Along the y-nullcline the vector field points left or right.

Return

Competing Species

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- The x-nullcline consists of the two lines $x = 0$ and $2x + y = 5$.
- The y-nullcline consists of the two lines $y = 0$ and $2x + 3y = 7$.
- The nullclines intersect at the equilibrium points.

Return Nullclines

- Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant $\to (2, 1)$ as $t \to \infty$.

Return

Competing Species – 2^{nd} Example

$$x' = (1 - x - y)x$$
$$y' = (4 - 7x - 3y)y$$

- The equilibrium point at $(1/4, 3/4)$ is a saddle point.
- All solutions go to either $(0, 4/3)$ or $(1, 0)$.

Return

Definition: The *basin of attraction* of a sink $\mathbf{y}_0$ consists of all points $\mathbf{y}$ such that the solution staring at $\mathbf{y}$ approaches $\mathbf{y}_0$ as $t \to \infty$.

- In the example , the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called *separatrices*. (Separatrices is the plural of separatrix.)

Return