

Math 211

Lecture #38

Linearization in Higher Dimension

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Linearization of a Planar System

$$x' = f(x, y)$$

$$y' = g(x, y)$$

- (x_0, y_0) is an eq. point so $f(x_0, y_0) = g(x_0, y_0) = 0$.
- Linearization at (x_0, y_0)

$$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \tilde{v}$$

$$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0) \cdot \tilde{v}$$

Matrix Form of the Linearization

Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

Theorem: Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the **linearization** at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
 - ◆ Occupy large open subsets of the trace-determinant plane.
- Nongeneric types: center and others.
 - ◆ Occupy pieces of the boundaries between the generic points.
 - ◆ Example:

$$x' = y + \alpha x(x^2 + y^2)$$

$$y' = -x + \alpha y(x^2 + y^2)$$

Higher Dimensional Systems

Autonomous equation $\mathbf{y}' = \mathbf{f}(\mathbf{y})$

- $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$
- $\mathbf{f}(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \dots, f_n(\mathbf{y}))^T$
- J is the **Jacobian matrix**

$$\mathbf{f}(\mathbf{y}_0 + \mathbf{u}) = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u}) \quad \text{where} \quad \lim_{\mathbf{u} \rightarrow \mathbf{0}} \frac{\mathbf{R}(\mathbf{u})}{|\mathbf{u}|} = \mathbf{0}.$$

- Set $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$. The system becomes

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u}).$$

- The linearization is $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$.

Theorem: Suppose that \mathbf{y}_0 is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of \mathbf{f} at \mathbf{y}_0 .

1. Suppose that the real part of every eigenvalue of J is negative. Then \mathbf{y}_0 is an asymptotically stable equilibrium point.
2. Suppose that J has at least one eigenvalue with positive real part. Then \mathbf{y}_0 is an unstable equilibrium point.

Example

$$x' = -2x - 4y + 2xy$$

$$y' = x - 6y + x^2 - y^2$$

- The origin $(0, 0)$ is an equilibrium point.
- The Jacobian has one eigenvalue, $\lambda = -4$, of algebraic multiplicity 2.
- **First theorem** does not apply.
- **Second theorem** \Rightarrow the origin is a sink.

The Lorenz System

$$x' = -ax + ay$$

$$y' = rx - y - xz$$

$$z' = -bz + xy$$

- Equilibrium points.
 - ◆ $(r \leq 1)$ $(0, 0, 0)$
 - ◆ $(r > 1)$ Set $s = \sqrt{b(r - 1)}$. The equilibrium points are $(0, 0, 0)$, and $\mathbf{c}^{\pm} = (\pm s, \pm s, r - 1)$.

- The Jacobian is

$$J = \begin{pmatrix} -a & a & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

- ♦ Use $a = 10$ and $b = 8/3$.
- ♦ $(0, 0, 0)$
 - ▶ If $r < 1$ $(0, 0, 0)$ is asymptotically stable.
 - ▶ If $r > 1$ $(0, 0, 0)$ is unstable.

- ◆ c^+ and c^-
 - ▶ For $1 < r < 470/19 \approx 24.74$, c^+ and c^- are asymptotically stable.
 - ▶ For $r > 470/19 \approx 24.74$, c^+ and c^- are unstable.

- ◆ As r varies the **Lorenz system** displays a wide variety of behaviors.
 - ▶ For $r = 28$ we have Lorenz's strange attractor.
 - ▶ For $r = 100$ there is a periodic attractor.
 - ▶ For $r = 200$ there is another strange attractor.

Invariant Sets

Definition: A set S is *(positively) invariant* for the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{y}(0) = \mathbf{y}_0 \in S$ implies that $\mathbf{y}(t) \in S$ for all $t \geq 0$.

- Examples:
 - ◆ An equilibrium point.
 - ◆ Any solution curve.

Example — Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

- The positive x - and y -axes are **invariant**.
- The positive quadrant is invariant.
 - ♦ Populations should remain nonnegative.
- The set $S = \{(x, y) \mid 0 < x < 3, 0 < y < 3\}$ is positively invariant.

Nullclines

$$x' = f(x, y)$$

$$y' = g(x, y)$$

Definition: The *x-nullcline* is the set defined by $f(x, y) = 0$. The *y-nullcline* is the set defined by $g(x, y) = 0$.

- Along the *x*-nullcline the vector field points up or down.
- Along the *y*-nullcline the vector field points left or right.

Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

- The x -nullcline consists of the two lines $x = 0$ and $2x + y = 5$.
- The y -nullcline consists of the two lines $y = 0$ and $2x + 3y = 7$.
- The nullclines intersect at the equilibrium points.

- Two of the **four regions** in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant $\rightarrow (2, 1)$ as $t \rightarrow \infty$.

Competing Species – 2nd Example

$$x' = (1 - x - y)x$$

$$y' = (4 - 7x - 3y)y$$

- The equilibrium point at $(1/4, 3/4)$ is a saddle point.
- All solutions go to either $(0, 4/3)$ or $(1, 0)$.

Definition: The *basin of attraction* of a sink y_0 consists of all points y such that the solution starting at y approaches y_0 as $t \rightarrow \infty$.

- In the *example*, the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called *separatrices*. (Separatrices is the plural of separatrix.)