Math 211

Lecture #39

Invariant Sets

November 30, 2001

Review of Methods

Linearization at an equilibrium point

- $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ has an equilibrium point at $\mathbf{y}_0$.
- The linearization $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$ has an equilibrium point at $\mathbf{u} = \mathbf{0}$.
- The linearization can sometimes predict the behavior of solutions to the nonlinear system *near the equilibrium point*.
- The linearization gives only local information.

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Theorem: Consider the planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Return Review

Theorem: Suppose that $\mathbf{y}_0$ is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of $\mathbf{f}$ at $\mathbf{y}_0$.

1. Suppose that the real part of every eigenvalue of J is negative. Then $\mathbf{y}_0$ is an asymptotically stable equilibrium point.
2. Suppose that J has at least one eigenvalue with positive real part. Then $\mathbf{y}_0$ is an unstable equilibrium point.

Invariant Sets

Definition: A set S is *(positively) invariant* for the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{y}(0) = \mathbf{y}_0 \in S$ implies that $\mathbf{y}(t) \in S$ for all $t \geq 0$.

- Examples:
  - An equilibrium point.
  - Any solution curve.

Example — Competing Species

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- The positive x- and y-axes are invariant.
- The positive quadrant is invariant.
  - Populations should remain nonnegative.
- The set $S = \{(x, y) \,|\, 0 < x < 3,\ 0 < y < 3\}$ is positively invariant.

Nullclines

$$x' = f(x, y)$$
$$y' = g(x, y)$$

Definition: The x-*nullcline* is the set defined by $f(x, y) = 0$. The y-*nullcline* is the set defined by $g(x, y) = 0$.

- Along the x-nullcline the vector field points up or down.
- Along the y-nullcline the vector field points left or right.

Competing Species

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- The x-nullcline consists of the two lines $x = 0$ and $2x + y = 5$.
- The y-nullcline consists of the two lines $y = 0$ and $2x + 3y = 7$.
- The nullclines intersect at the equilibrium points.

- Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant $\to (2, 1)$ as $t \to \infty$.

Competing Species – 2^{nd} Example

$$x' = (1 - x - y)x$$
$$y' = (4 - 7x - 3y)y$$

- The axes are invariant. The positive quadrant is invariant.
- The equilibrium point at $(1/4, 3/4)$ is a saddle point.
- Almost all solutions go to one of the nodal sinks $(0, 4/3)$ or $(1, 0)$.

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Definition: The *basin of attraction* of a sink $\mathbf{y}_0$ consists of all points $\mathbf{y}$ such that the solution staring at $\mathbf{y}$ approaches $\mathbf{y}_0$ as $t \to \infty$.

- In the example , the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called *separatrices*. (Separatrices is the plural of separatrix.)

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Summary

- Sometimes the understanding of invariant sets can help us understand the long term behavior of all solutions.
- Nullclines can sometimes help us find informative invariant sets.
- None of this helps us understand the predator-prey system.