

Math 211

Lecture #39

Invariant Sets

November 30, 2001

Review of Methods

Linearization at an equilibrium point

- $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ has an equilibrium point at \mathbf{y}_0 .
- The linearization $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$ has an equilibrium point at $\mathbf{u} = \mathbf{0}$.
- The linearization can sometimes predict the behavior of solutions to the nonlinear system *near the equilibrium point*.
- The linearization gives only local information.

Theorem: Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Theorem: Suppose that \mathbf{y}_0 is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of \mathbf{f} at \mathbf{y}_0 .

1. Suppose that the real part of every eigenvalue of J is negative. Then \mathbf{y}_0 is an asymptotically stable equilibrium point.
2. Suppose that J has at least one eigenvalue with positive real part. Then \mathbf{y}_0 is an unstable equilibrium point.

Invariant Sets

Definition: A set S is *(positively) invariant* for the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{y}(0) = \mathbf{y}_0 \in S$ implies that $\mathbf{y}(t) \in S$ for all $t \geq 0$.

- Examples:
 - ◆ An equilibrium point.
 - ◆ Any solution curve.

Example — Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

- The positive x - and y -axes are **invariant**.
- The positive quadrant is invariant.
 - ◆ Populations should remain nonnegative.
- The set $S = \{(x, y) \mid 0 < x < 3, 0 < y < 3\}$ is positively invariant.

Nullclines

$$x' = f(x, y)$$

$$y' = g(x, y)$$

Definition: The *x-nullcline* is the set defined by $f(x, y) = 0$. The *y-nullcline* is the set defined by $g(x, y) = 0$.

- Along the *x*-nullcline the vector field points up or down.
- Along the *y*-nullcline the vector field points left or right.

Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

- The x -nullcline consists of the two lines $x = 0$ and $2x + y = 5$.
- The y -nullcline consists of the two lines $y = 0$ and $2x + 3y = 7$.
- The nullclines intersect at the equilibrium points.

- Two of the **four regions** in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant $\rightarrow (2, 1)$ as $t \rightarrow \infty$.

Competing Species – 2nd Example

$$x' = (1 - x - y)x$$

$$y' = (4 - 7x - 3y)y$$

- The axes are invariant. The positive quadrant is invariant.
- The equilibrium point at $(1/4, 3/4)$ is a saddle point.
- Almost all solutions go to one of the nodal sinks $(0, 4/3)$ or $(1, 0)$.

Definition: The *basin of attraction* of a sink y_0 consists of all points y such that the solution starting at y approaches y_0 as $t \rightarrow \infty$.

- In the *example*, the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called *separatrices*. (Separatrices is the plural of separatrix.)

Summary

- Sometimes the understanding of invariant sets can help us understand the long term behavior of all solutions.
- Nullclines can sometimes help us find informative invariant sets.
- None of this helps us understand the predator-prey system.