Math 211

Lecture #40

Long Term Behavior of Planar Systems

December 3, 2001

Basic Question about a System $\mathbf{y}' = \mathbf{f}(\mathbf{y})$

- What happens to all solutions as $t \to \infty$?
- What are the possibilities as $t \to \infty$?
  - Is there a small list of all possibilities?
  - We need a definitive notion of what a “possibility” is.

Limit Sets

Definition: The (forward) limit set of the solution $\mathbf{y}(t)$ that starts at $\mathbf{y}_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.

- $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \to \infty$ such that $\mathbf{y}(t_k) \to \mathbf{x}$.
- What kinds of sets can be limit sets?
  - The empty set.
  - Equilibrium points.
  - Periodic solution curves.
- Is there a small list of all possibile limit sets?

Properties of Limit Sets

Theorem: Suppose that the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ is defined in the set U.

1. If the solution curve starting at $\mathbf{y}_0$ stays in a bounded subset of U, then the limit set $\omega(\mathbf{y}_0)$ is not empty.
2. Any limit set is both positively and negatively invariant.

Example

$$x' = 5y + x(9 - x^2 - y^2)$$
$$y' = -5x + y(9 - x^2 - y^2)$$

- The origin is a spiral source.
- In polar coordinates the system is

$$r' = r(9 - r^2)$$
$$\theta' = -5$$

- All solution curves approach the circle $x^2 + y^2 = 9$.
  - The circle $x^2 + y^2 = 9$ is a solution curve.

Limit Cycle

Definition: A limit cycle is a closed solution curve which is the limit set of nearby solution curves. If the solution curves spiral into the limit cycle as $t \longrightarrow \infty$, it is a attracting limit cycle. If they spiral into the limit cycle as $t \longrightarrow -\infty$, it is a repelling limit cycle.

- In the example the circle $x^2 + y^2 = 9$ is a limit cycle.

Return

Types of Limit Set

- A limit cycle is a new type of phenomenon.
- However, the limit set is a periodic orbit, so the type of limit set is not new.
- We still have only two types of non-empty limits sets.
  - An equilibirum point.
  - A closed solution curve.

Example

$$x' = (y + x/5)(1 - x^2)$$
$$y' = -x(1 - y^2)$$

- The lines $x = \pm 1$ and $y = \pm 1$ are invariant.
- The unit square is invariant.
- The corners of the unit square are saddle points.
  - The lines $x = \pm 1$ and $y = \pm 1$ are separatrices.
- The origin is a spiral source.
- The limit set of any solution that starts in the unit square is the boundary of the unit square.

Planar Graph

Definition: A *planar graph* is a collection of points, called *vertices*, and non-intersecting curves, called *edges*, which connect the vertices. If the edges each have a direction the graph is said to be *directed*.

- The boundary of the unit square in the example is a directed planar graph.

Theorem: If S is a nonempty limit set of a solution of a planar system defined in a set $U \subset \mathbf{R}^2$, then S is one of the following:

- An equilibrium point.
- A closed solution curve.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the *Poincaré-Bendixson alternatives.*

Remarks

- These three are the only possibilities.
- The closed solution curve could be a limit cycle.
- If a vertex of a limiting planar graph is a generic equilibrium point, then it must be a saddle point. The edges connecting this point must be separatrices.

Poincaré-Bendixson Theorem

Theorem: Suppose that R is a closed and bounded planar region that is positively invariant for a planar system. If R contains no equilibrium points, then there is a closed solution curve in R.

- The theorem is true if the set R is negatively invariant.

Examples

- # 1.

$$x' = x + y - x(x^2 + 3y^2)$$
$$y' = -x + y - 2y^3$$

  - The set $\{(x, y) \,|\, 0.5 \leq x^2 + y^2 \leq 1\}$ is positively invariant.

- # 2. Rayleigh's example: $z'' + \mu z'[(z')^2 - 1] + z = 0.$