

Math 211

Lecture #41

Limit Cycles and the Pendulum

December 5, 2001

Basic Question about $\mathbf{y}' = \mathbf{f}(\mathbf{y})$

- The (forward) limit set of the solution $\mathbf{y}(t)$ that starts at \mathbf{y}_0 is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.
 - ♦ $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \rightarrow \infty$ such that $\mathbf{y}(t_k) \rightarrow \mathbf{x}$.
- What is $\omega(\mathbf{y}_0)$ for all \mathbf{y}_0 ?

[Return](#)

Theorem: If S is a nonempty limit set of a solution of a planar system defined in a set $U \subset \mathbf{R}^2$, then S is one of the following:

- An equilibrium point.
- A closed solution curve.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the *Poincaré-Bendixson alternatives*.

[Return](#)

Poincaré-Bendixson Theorem

Theorem: Suppose that R is a closed and bounded planar region that is positively invariant for a planar system. If R contains no equilibrium points, then there is a closed solution curve in R .

- The theorem is true if the set R is negatively invariant.

[Return](#)

[Poincaré-Bendixson alternatives](#)

Examples

- # 1.

$$x' = x + y - x(x^2 + 3y^2)$$

$$y' = -x + y - 2y^3$$

- The set $\{(x, y) \mid 0.5 \leq x^2 + y^2 \leq 1\}$ is positively invariant.
- # 2. Rayleigh's example: $z'' + \mu z'[(z')^2 - 1] + z = 0$.
 - There is a limit cycle.

The Pendulum

- The angle θ satisfies the nonlinear differential equation

$$mL\theta'' = -mg \sin \theta - D \theta',$$

or

$$\theta'' + \frac{D}{mL}\theta' + \frac{g}{L} \sin \theta = 0.$$

- We will write this as

$$\theta'' + d\theta + b \sin \theta = 0.$$

[Return](#)

The Pendulum System

- Introduce $\omega = \theta'$ to get the system

$$\theta' = \omega$$

$$\omega' = -b \sin \theta - d\omega$$

- The equilibrium points are $(k\pi, 0)^T$ where k is any integer.
 - ♦ If k is odd the equilibrium point is a saddle.
 - ♦ If k is even the equilibrium point is a center if $d = 0$ or a sink if $d > 0$.

[Return](#)

[Pendulum](#)

The Inverted Pendulum

- The angle θ measured from straight up satisfies the nonlinear differential equation

$$mL\theta'' = mg \sin \theta - D\theta',$$

or

$$\theta'' + \frac{D}{mL}\theta' - \frac{g}{L}\sin \theta = 0.$$

- ♦ We will write this as

$$\theta'' + d\theta - b \sin \theta = 0.$$

[Return](#)

[Pendulum](#)

The Inverted Pendulum System

- Introduce $\omega = \theta'$ to get the system

$$\theta' = \omega$$

$$\omega' = b \sin \theta - d\omega$$

- The equilibrium point at $(0, 0)^T$ is a saddle point and unstable.
- Can we find an automatic way of sensing the departure of the system from $(0, 0)^T$ and moving the pivot to bring the system back to the unstable point at $(0, 0)^T$?
 - ♦ Experimentally the answer is yes.

[Return](#)

[Inverted pendulum](#)

[Pendulum system](#)

The Control System

- If we apply a force v moving the pivot to the right or left, then θ satisfies

$$mL\theta'' = mg \sin \theta - D\theta' - v \cos \theta,$$

- The system becomes

$$\theta' = \omega$$

$$\omega' = b \sin \theta - d\omega - u \cos \theta,$$

where $u = v/mL$.

- The force is a linear response to the detected values of θ and ω , so $u = c_1\theta + c_2\omega$, where c_1 and c_2 are constants.

[Return](#) [Inverted pendulum](#) [Inverted pendulum system](#)

The Controlled System

- The Jacobian at the origin is

$$J = \begin{pmatrix} 0 & 1 \\ b - c_1 & -d - c_2 \end{pmatrix}$$

- The origin is asymptotically stable if $T = -(d + c_2) < 0$ and $D = c_1 - b > 0$. Therefore require

$$c_1 > b = \frac{g}{L} \quad \text{and} \quad c_2 > -d = -\frac{D}{mL}.$$

[Return](#) [Inverted pendulum](#) [Inverted pendulum system](#) [Controls](#)