# Math 211

Lecture #41

Limit Cycles and the Pendulum

December 5, 2001

# Basic Question about $\mathbf{y}' = \mathbf{f}(\mathbf{y})$

- The (forward) limit set of the solution $\mathbf{y}(t)$ that starts at $\mathbf{y}_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.
  - $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \to \infty$ such that $\mathbf{y}(t_k) \to \mathbf{x}$.
- What is $\omega(\mathbf{y}_0)$ for all $\mathbf{y}_0$?

Return

**Theorem:** If $S$ is a nonempty limit set of a solution of a planar system defined in a set $U \subset \mathbf{R}^2$, then $S$ is one of the following:

- An equilibrium point.
- A closed solution curve.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the *Poincaré-Bendixson alternatives.*

# Poincaré-Bendixson Theorem

**Theorem:** Suppose that $R$ is a closed and bounded planar region that is positively invariant for a planar system. If $R$ contains no equilibrium points, then there is a closed solution curve in $R$.

- The theorem is true if the set $R$ is negatively invariant.

# Examples

- # 1.

$$x' = x + y - x(x^2 + 3y^2)$$
$$y' = -x + y - 2y^3$$

  - The set $\{(x, y) \,|\, 0.5 \leq x^2 + y^2 \leq 1\}$ is positively invariant.

- # 2. Rayleigh's example: $z'' + \mu z'[(z')^2 - 1] + z = 0.$
  - There is a limit cycle.

# The Pendulum

- The angle $\theta$ satisfies the nonlinear differential equation

$$mL\theta'' = -mg\,\sin\theta - D\,\theta',$$

or

$$\theta'' + \frac{D}{mL}\theta' + \frac{g}{L}\sin\theta = 0.$$

  - We will write this as

$$\theta'' + d\,\theta + b\,\sin\theta = 0.$$

# The Pendulum System

- Introduce $\omega = \theta'$ to get the system

$$\begin{aligned} \theta' &= \omega \\ \omega' &= -b\,\sin\theta - d\,\omega \end{aligned}$$

- The equilibrium points are $(k\,\pi, 0)^T$ where $k$ is any integer.
  - If $k$ is odd the equilibrium point is a saddle.
  - If $k$ is even the equilibrium point is a center if $d = 0$ or a sink if $d > 0$.

# The Inverted Pendulum

- The angle $\theta$ measured from straight up satisfies the nonlinear differential equation

$$mL\theta'' = mg\,\sin\theta - D\,\theta',$$

or

$$\theta'' + \frac{D}{mL}\theta' - \frac{g}{L}\sin\theta = 0.$$

  - We will write this as

$$\theta'' + d\,\theta - b\,\sin\theta = 0.$$

# The Inverted Pendulum System

- Introduce $\omega = \theta'$ to get the system

$$\theta' = \omega$$
$$\omega' = b \sin\theta - d\,\omega$$

- The equilibrium point at $(0,0)^T$ is a saddle point and unstable.
- Can we find an automatic way of sensing the departure of the system from $(0,0)^T$ and moving the pivot to bring the system back to the unstable point at $(0,0)^T$?
  - Experimentally the answer is yes.

# The Control System

- If we apply a force $v$ moving the pivot to the right or left, then $\theta$ satisfies

$$mL\theta'' = mg\,\sin\theta - D\,\theta' - v\,\cos\theta,$$

- The system becomes

$$\theta' = \omega$$
$$\omega' = b\,\sin\theta - d\,\omega - u\,\cos\theta,$$

where $u = v/mL$.

- The force is a linear response to the detected values of $\theta$ and $\omega$, so $u = c_1\theta + c_2\omega$, where $c_1$ and $c_2$ are constants.

# The Controlled System

- The Jacobian at the origin is

$$J = \begin{pmatrix} 0 & 1 \\ b - c_1 & -d - c_2 \end{pmatrix}$$

- The origin is asymptotically stable if $T = -(d + c_2) < 0$ and $D = c_1 - b > 0$. Therefore require

$$c_1 > b = \frac{g}{L} \quad \text{and} \quad c_2 > -d = -\frac{D}{mL}.$$