Math 211

Review for the Final Exam

December 8, 2002

The Final Exam

- The final will be comprehensive, covering material from the entire semester.
- The final will emphasize the material covered since the last exam.
- These slides will cover primarily the material covered since the last exam. They do *not* cover all of the material on the exam.
- Questions about any of the material of the course will be answered.

The Themes of the Course

- Modeling.
  - Population, finance, mixing, motion, vibrating spring, electrical circuits, . . .
- Exact solutions.
  - Separable and linear equations in dimension 1.
  - Linear equations in higher dimension.
    - Matrix algebra.
  - Second order equations.
- Numerical solutions.
- Geometric analysis.

Solving $\mathbf{x}' = A\mathbf{x}$

- A is an $n \times n$ matrix.
- Solution strategy: Look for a fundamental set of solutions, i.e., n linearly independent solutions.
- The function $\mathbf{x}(t) = e^{tA}\mathbf{v}$ solves the initial value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{v}$.
- Refined strategy: Compute $e^{tA}\mathbf{v}$ for n linearly independent vectors $\mathbf{v}$.
  - Computing $e^{tA}\mathbf{v}$ is hard except for specially chosen vectors $\mathbf{v}$.

Key to Computing $e^{tA}\mathbf{v}$

Suppose that A an $n \times n$ matrix, and λ a number (an eigenvalue). Then

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda t} \cdot e^{t(A-\lambda I)}\mathbf{v} \\ &= e^{\lambda t} \cdot \left[\mathbf{v} + t(A-\lambda I)\mathbf{v} + \frac{t^2}{2!}(A-\lambda I)^2\mathbf{v} + \cdots\right] \end{aligned}$$

- If λ is an eigenvalue and $\mathbf{v}$ is an associated eigenvector, then $(A-\lambda I)\mathbf{v} = \mathbf{0}$, so $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
- If $(A-\lambda I)^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}[\mathbf{v} + t(A-\lambda I)\mathbf{v}]$.

Generalized Eigenvectors

Definition: If λ is an eigenvalue of A and $(A - \lambda I)^p \mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$, then $\mathbf{v}$ is called a *generalized eigenvector* associated with λ.

- Then

$$e^{tA}\mathbf{v} = e^{\lambda t}\Big[\mathbf{v} + +t(A - \lambda I)\mathbf{v} + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v} + \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{v}\Big]$$

- We can compute $e^{tA}\mathbf{v}$ for any generalized eigenvector.

Multiplicities

A an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.

- The characteristic polynomial has the form

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of λ_j is q_j.
  - $q_1 + q_2 + \ldots + q_k = n$.
- The *geometric multiplicity* of λ_j is d_j, the dimension of the eigenspace of λ_j.
  - $1 \le d_j \le q_j$.
- There is an integer $k_j \le q_j$ for which $\mathrm{null}((A - \lambda_j I)^{k_j})$ has dimension q_j.

Procedure for Solving $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A and their algebraic multiplicities.
- For each eigenvalue λ with algebraic multiplicity q:
  - Find the smallest integer k for which $\mathrm{null}((A - \lambda I)^k)$ has dimension q.
  - Find a basis for $\mathrm{null}((A - \lambda I)^k)$.
  - For each vector $\mathbf{v}$ in the basis compute the solution $\mathbf{x}(t) = e^{tA}\mathbf{v}$.
- The set of all of these solutions is a fundamental set of solutions.

Replacing Complex Solutions with Real Solutions

- If A has complex eigenvalues, the fundamental set of solutions contains complex valued solutions.
- Complex solutions occur in complex conjugate pairs $\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t)$ and $\overline{\mathbf{z}(t)} = \mathbf{x}(t) - i\mathbf{y}(t)$.
- Replace $\mathbf{z}(t)$ and $\overline{\mathbf{z}(t)}$ with the real solutions $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t))$.

Solutions to Higher Order Equations

Homogenous linear equation with constant coefficients:

$$y'' + py' + qy = 0$$

- Look for exponential solutions $y(t) = e^{\lambda t}$.
- *Characteristic polynomial:* $\lambda^2 + p\lambda + q$.
- If λ is a root of the characteristic polynomial then $y(t) = e^{\lambda t}$ is a solution.

Fundamental sets of solutions

- Two distinct real roots λ_1 and λ_2:

$$y(_1t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t}.$$

- One real root λ of multiplicity 2:

$$y_1(t) = e^{\lambda t} \quad \text{and} \quad y_2(t) = te^{\lambda t}.$$

- Complex conjugate roots $\lambda = \alpha \pm i\beta$:

$$y_1(t) = e^{\alpha t}\cos\beta t \quad \text{and} \quad y_2(t) = e^{\alpha t}\sin\beta t.$$

Inhomogeneous Equations

$$y'' + Py' + Qy = f(t)$$

- The method of undetermined coefficients finds a particular solution $y_p(t)$.
- The general solution is

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t),$$

  where y_1 and y_2 are a fundamental set of solutins to the homogeneous equation.
- If the forcing term $f(t)$ has a form which is replicated under differentiation, look for a particular solution of the same general form as the forcing term.

Cases

- If $f(t) = Ce^{bt}$, try $y_p(t) = ae^{bt}$.
- If $f(t) = A\cos\omega t + B\sin\omega t$, try $y_p(t) = a\cos\omega t + b\sin\omega t$.
  - Or try the complex method.
- If $f(t)$ is a polynomial of degree n, let y_p be a polynomial of degree n.
- Exceptional cases: Multiply expected form of y_p by t.
- Combination cases: Solve the equation in pieces.

Harmonic Motion

- Spring: $y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t)$.
- Circuit: $I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{1}{L} E'(t)$.
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

  - We call this the equation for *harmonic motion.*
- ω_0 is the *natural frequency.* c is the *damping constant.* $f(t)$ is the *forcing term.*

Unforced Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = 0$$

- Undamped: $c = 0$.
- Underdamped: $0 < c < \omega_0$.
- Critically damped: $c = \omega_0$.
- Over damped: $c > \omega_0$.

Forced Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t$$

- A is the *forcing amplitude* and ω is the *forcing frequency.*
- The general solution is $x(t) = x_p(t) + x_h(t)$.
  - x_p is a particular solution. x_h is the general solution of the homogenous equation.
- Undamped: $c = 0$.
  - $\omega \neq \omega_0$: Beats.
  - $\omega = \omega_0$: Resonance.

Forced, Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

- $c > 0$ implies that $x_h(t) \to 0$ as t increases, so x_h is called the *transient term*.
- $x_p(t)$ is called the *steady-state solution.* It has the form

$$x_p(t) = G(\omega)A\cos(\omega t - \phi)$$

  - x_p is oscillatory at the driving frequency.
  - The amplitude of x_p is the product of the *gain*, $G(\omega)$, and the amplitude of the forcing function.
  - x_p has a *phase shift* of ϕ with respect to the forcing function.

Qualitative Analysis

- Existence and uniqueness.
- For an autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, the basic question is, What happens to *all solutions* as $t \to \infty$?
- The easy cases: equilibrium points $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$ and equilibrium solutions $\mathbf{x}(t) = \mathbf{x}_0$.
- Local qualitative analysis: What happens as $t \to \infty$ to all solutions that start near an equilibrium point $\mathbf{x}_0$?
  - This is the question of stability.
- Global qualitative analysis: What happens to *all solutions* as $t \to \infty$?

Stability

Suppose the autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ has an equilibrium point at $\mathbf{x}_0$.

- $\mathbf{x}_0$ is *stable* if every solution that starts close to $\mathbf{x}_0$ stays close to $\mathbf{x}_0$.
- $\mathbf{x}_0$ is *asymptotically stable* if every solution that starts close to $\mathbf{x}_0$ stays near $\mathbf{x}_0$ and approaches $\mathbf{x}_0$ as $t \to \infty$.
    - $\mathbf{x}_0$ is called a *sink.*
- $\mathbf{x}_0$ is *unstable* if there are solutions starting arbitrarily close to $\mathbf{x}_0$ that move away from $\mathbf{x}_0$.

Stability for $\mathbf{x}' = A\mathbf{x}$

- $D = 2$: Trace-determinant plane.
- **Theorem:** Let A be an $n \times n$ real matrix.
  - Suppose the real part of every eigenvalue of A is negative. Then $\mathbf{0}$ is an asymptotically stable equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.
  - Suppose A has at least one eigenvalue with positive real part. Then $\mathbf{0}$ is an unstable equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.

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Stability for $\mathbf{x}' = \mathbf{f}(\mathbf{x})$

- Suppose that $\mathbf{x}_0$ is an equilibrium point.
- The *linearization* at $\mathbf{x}_0$ is the system $\mathbf{u}' = J\mathbf{u}$, where J is the *Jacobian matrix* of $\mathbf{f}$ at $\mathbf{x}_0$.
- For the planar system $\left\{ \begin{array}{l} x' = f(x,y) \\ y' = g(x,y) \end{array} \right\}$, the Jacobian is

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0, y_0) & \dfrac{\partial f}{\partial y}(x_0, y_0) \\ \dfrac{\partial g}{\partial x}(x_0, y_0) & \dfrac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

Stability for $D = 2$

Theorem: Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Stability for $D \geq 1$

Theorem: Suppose that $\mathbf{y}_0$ is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of $\mathbf{f}$ at $\mathbf{y}_0$.

1. Suppose that the real part of every eigenvalue of J is negative. Then $\mathbf{y}_0$ is an asymptotically stable equilibrium point.

2. Suppose that J has at least one eigenvalue with positive real part. Then $\mathbf{y}_0$ is an unstable equilibrium point.

Global Geometric Analysis

- What happens to *all solutions* as $t \to \infty$?
- The (forward) limit set of the solution $\mathbf{y}(t)$ that starts at $\mathbf{y}_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.
  - $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \to \infty$ such that $\mathbf{y}(t_k) \to \mathbf{x}$.
- What is $\omega(\mathbf{y}_0)$ for all $\mathbf{y}_0$?
  - What is the limit set for all solutions?
- In dimension 1, all limit sets are equilibrium points.

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Limit Sets in Dimension 2

Theorem: If S is a nonempty limit set of a solution of a planar system defined in a set $U \subset \mathbf{R}^2$, then S is one of the following:

- An equilibrium point.
- A closed solution curve.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the *Poincaré-Bendixson alternatives.*

- In dimension 3 the answer is unknown.

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Invariant Sets

Definition: A set S is *(positively) invariant* for the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{y}(0) = \mathbf{y}_0 \in S$ implies that $\mathbf{y}(t) \in S$ for all $t \geq 0$.

- Examples include equilibrium points, and any solution curve.
- In dimension 2, invariant sets can frequently be found using:
  - nullclines,
  - polar coordinants.

Poincaré-Bendixson Theorem

Theorem: Suppose that R is a closed and bounded planar region that is positively invariant for a planar system. If R contains no equilibrium points, then there is a closed solution curve in R.

- The theorem is also true if the set R is negatively invariant.
- The closed solution curve might be a limit cycle.

Solving Separable Equations

$$\frac{dy}{dt} = g(y)h(t)$$

The three step solution process:

1. Separate the variables. $\dfrac{dy}{g(y)} = h(t)\,dt$ if $g(y) \neq 0$.
2. Integrate both sides. $\displaystyle\int \frac{dy}{g(y)} = \int h(t)\,dt$
3. Solve for $y(t)$.

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Solving the Linear Equation

$$x' = a(t)x + f(t)$$

Four step process:

1. Rewrite as $x' - ax = f$.
2. Multiply by the integrating factor

$$u(t) = e^{-\int a(t)\,dt}.$$

   Equation becomes $[ux]' = ux' - aux = uf$.

3. Integrate: $u(t)x(t) = \int u(t)f(t)\,dt + C$.
4. Solve for $x(t)$.

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Eigenvalues and Eigenvectors

- λ is an *eigenvalue* of A if there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. If λ is an eigenvalue of A, then any vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with* λ.
- λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$.
  - $p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial* of A.
- $\mathbf{v}$ is an eigenvector associated with the eigenvalue $\lambda \Leftrightarrow \mathbf{v} \in \mathrm{null}(A - \lambda I)$.
  - $\mathrm{null}(A - \lambda I)$ is called the *eigenspace* of λ.