Math 211

Lecture #5

Models of Motion

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Models of Motion

History of models of planetary motion.

- Babylonians - 3000 years ago.
  - Initiated the systematic study of astronomy.
  - Collection of astonomical data.

Greeks

- Descriptive model - Ptolemy (~ 100).
  - Geocentric model.
  - Epicycles.
- Enabled predictions.
- Provided no causal explanation.
- This model was refined over the following 1400 years.

Nicholas Copernicus (1543)

- Shifted the center of the universe to the sun.
- Fewer epicycles required.
- Still descriptive and provided no causal explanation.
- The shift to a sun centered universe was a major change in human understanding of their place in the universe.

Johann Kepler (1609)

- Based on experimental work of Tycho Brahe (1400).
- Three laws of planetary motion.
  1. Each planet moves in an ellipse with the sun at one focus.
  2. The line between the sun and a planet sweeps out equal areas in equal times.
  3. The ratio of the cube of the semi-major axis to the square of the period is the same for each planet.
- This model was still descriptive and not causal.

Isaac Newton

- Three major contributions.
  - Laws of mechanics.
    - Second law — $F = ma$.
  - Universal law of gravity.
  - Fundamental theorem of calculus.
    - $f' = g \Leftrightarrow \int g(x)\,dx = f(x) + C$.
    - Invention of calculus.
  - *Principia Mathematica* 1687

Isaac Newton (cont.)

- Laws of mechanics and gravitation were based on his own experiments and his understanding of the experiments of others.
- Derived Kepler's three laws of planetary motion.
- This was a causal explanation.
  - For any mechanical motion.
  - Still used today.

Isaac Newton (cont.)

- *The Life of Isaac Newton* by Richard Westfall, Cambridge University Press 1993.
- Problems with Newton's theory.
  - The force of gravity was action at a distance.
  - Physical anomalies.
    - The Michelson-Morley experiment (1881-87).

Albert Einstein

- Special theory of relativity – 1905.
- General theory of relativity – 1916.
  - Gravity is due to curvature of space-time.
  - Curvature of space-time is caused by mass.
  - Gravity is no longer action at a distance.
- All known anomalies explained.

Unified Theories

- Four fundamental forces.
  - Gravity, electromagnetism, strong nuclear, and weak nuclear.
- Last three can be unified by quantum mechanics. — Quantum chromodynamics.
- Currently there are attempts to include gravity.
  - String theory.
  - *The Elegant Universe : Superstrings, hidden dimensions, and the quest for the ultimate theory* by Brian Greene, W.W.Norton, New York 1999.

The Modeling Process

- It is based on experiment and/or observation.
- It is iterative.
  - For motion we have ≥ 6 iterations.
  - After each change in the model it must be checked by further experimentation and observation.
- It is rare that a model captures all aspects of the phenomenon.

Linear Motion

- Motion in one dimension — $x(t)$ is the distance from a reference position.
- Example: motion of a ball in the earth's gravity — $x(t)$ is the height of the ball above the surface of the earth.
- Velocity: $v = x'$
- Acceleration: $a = v' = x''$.

- Acceleration due to gravity is (approximately) constant near the surface of the earth

$$F = -mg, \quad \text{where} \quad g = 9.8m/s^2$$

- Newton's second law: $F = ma$
- Equation of motion: $ma = -mg$, which becomes

$$x'' = -g \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -g. \end{aligned}$$

- Solving the system
$$\begin{aligned} x' &= v, \\ v' &= -g \end{aligned}$$
- Integrate the second equation:
$$v(t) = -gt + c_1$$
- Substitute into the first equation and integate:
$$x(t) = -\frac{1}{2}gt^2 + c_1 t + c_2.$$

Return

Air Resistance

Acts in the direction opposite to the velocity. Therefore

$$R(x,v) = -r(x,v)v \quad \text{where} \quad r(x,v) \geq 0.$$

There are many models. We will look at two different cases.

1. The resistance is proportional to velocity.
2. The magnitude of the resistance is proportional to the square of the velocity.

Resistance Proportional to Velocity

- $R(x, v) = -rv$, r a positive constant.
- Total force: $F = -mg - rv$
- Newton's second law: $F = ma$
- Equation of motion:

$$mx'' = -mg - rv \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -\frac{mg + rv}{m}. \end{aligned}$$

- The equation $v' = -\dfrac{mg + rv}{m}$ is separable.

- Solution is $v(t) = Ce^{-rt/m} - \dfrac{mg}{r}$.
- Notice
$$\lim_{t \to \infty} v(t) = -\frac{mg}{r}.$$
- The *terminal velocity* is $v_{\mathrm{term}} = -\dfrac{mg}{r}$.

Magnitude of Resistance Proportional to the Square of the Velocity

- $R(x, v) = -k|v|v$, k a positive constant.
- Total force: $F = -mg - k|v|v$.
- Equation of motion:

$$mx'' = -mg - k|v|v \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -\frac{mg + k|v|v}{m}. \end{aligned}$$

- The equation for v is separable.

- Suppose a ball is dropped from a high point. Then $v < 0$.
- The equation is $v' = \dfrac{-mg + kv^2}{m}$.
- The solution is

$$v(t) = \sqrt{\frac{mg}{k}}\,\frac{Ae^{-2t\sqrt{kg/m}} - 1}{Ae^{-2t\sqrt{kg/m}} + 1}.$$

- The terminal velocity is

$$v_{\text{term}} = -\sqrt{mg/k}.$$

Solving for $x(t)$

- Integrating $x' = v(t)$ is sometimes hard.
- Use the trick (see Exercise 2.3.7):

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

- The equation

$$v \frac{dv}{dx} = a$$

is usually separable.

Problem

A ball is projected from the surface of the earth with velocity v_0. How high does it go?

- At $t = 0$, we have $x(0) = 0$ and $v(0) = v_0$.
- At the top we have $t = T$, $x(T) = x_{\max}$, and $v(T) = 0$.

$R = 0$

The acceleration is $a = -g$. The equation $v\dfrac{dv}{dx} = a$ becomes

$$\begin{aligned} v\,dv &= -g\,dx \\ \int_{v_0}^{0} v\,dv &= -\int_{0}^{x_{\max}} g\,dx \\ -\frac{v_0^2}{2} &= -g x_{\max} \\ x_{\max} &= \frac{v_0^2}{2g}. \end{aligned}$$

$R = -rv$

The acceleration is $a = -(mg + rv)/m$. The equation $v\dfrac{dv}{dx} = a$ becomes

$$\int_{v_0}^{0} \frac{v\,dv}{rv + mg} = -\int_{0}^{x_{\max}} \frac{dx}{m}.$$

Solving, we get

$$x_{\max} = \frac{m}{r}\left[v_0 - \frac{mg}{r}\ln\left(1 + \frac{rv_0}{mg}\right)\right].$$

$$R = -k|v|v$$

Since $v > 0$, the acceleration is $a = -\dfrac{mg + kv^2}{m}$. The equation $v\dfrac{dv}{dx} = a$ becomes

$$\int_{v_0}^{0} \frac{v\,dv}{kv^2 + mg} = -\int_0^{x_{\max}} \frac{dx}{m}.$$

Solving, we get

$$x_{\max} = \frac{m}{2k} \ln\left(1 + \frac{kv_0^2}{mg}\right).$$