Math 211

Lecture #5

Models of Motion

September 6, 2002

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  - Initiated the systematic study of astronomy.
  - Collection of astonomical data.

Greeks

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  - Geocentric model.

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  - Geocentric model.
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- Enabled predictions.
- Provided no causal explanation.
- This model was refined over the following 1400 years.

Return

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- Still descriptive and provided no causal explanation.
- The shift to a sun centered universe was a major change in human understanding of their place in the universe.

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  3. The ratio of the cube of the semi-major axis to the square of the period is the same for each planet.
- This model was still descriptive and not causal.

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  - *Principia Mathematica* 1687

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  - For any mechanical motion.
  - Still used today.

Isaac Newton (cont.)

- *The Life of Isaac Newton* by Richard Westfall, Cambridge University Press 1993.

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- *The Life of Isaac Newton* by Richard Westfall, Cambridge University Press 1993.
- Problems with Newton's theory.
  - The force of gravity was action at a distance.
  - Physical anomalies.
    - The Michelson-Morley experiment (1881-87).

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- All known anomalies explained.

Unified Theories

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- Last three can be unified by quantum mechanics. — Quantum chromodynamics.
- Currently there are attempts to include gravity.
  - String theory.
  - *The Elegant Universe : Superstrings, hidden dimensions, and the quest for the ultimate theory* by Brian Greene, W.W.Norton, New York 1999.

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  - For motion we have ≥ 6 iterations.
  - After each change in the model it must be checked by further experimentation and observation.
- It is rare that a model captures all aspects of the phenomenon.

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Linear Motion

- Motion in one dimension — $x(t)$ is the distance from a reference position.
- Example: motion of a ball in the earth's gravity — $x(t)$ is the height of the ball above the surface of the earth.
- Velocity: $v = x'$
- Acceleration: $a = v' = x''$.

- Acceleration due to gravity is (approximately) constant near the surface of the earth

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$$x'' = -g \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -g. \end{aligned}$$

- Solving the system

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- Integrate the second equation:
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1. The resistance is proportional to velocity.
2. The magnitude of the resistance is proportional to the square of the velocity.

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- The equation $v' = -\dfrac{mg + rv}{m}$ is separable.

- Solution is $v(t) = Ce^{-rt/m} - \dfrac{mg}{r}$.

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- The *terminal velocity* is $v_{\mathrm{term}} = -\dfrac{mg}{r}$.

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- The equation for v is separable.

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- The solution is

$$v(t) = \sqrt{\frac{mg}{k}}\,\frac{Ae^{-2t\sqrt{kg/m}} - 1}{Ae^{-2t\sqrt{kg/m}} + 1}.$$

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Solving for $x(t)$

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- Use the trick (see Exercise 2.3.7):

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

- The equation

$$v \frac{dv}{dx} = a$$

is usually separable.

Problem

A ball is projected from the surface of the earth with velocity v_0. How high does it go?

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- At $t = 0$, we have $x(0) = 0$ and $v(0) = v_0$.
- At the top we have $t = T$, $x(T) = x_{\max}$, and $v(T) = 0$.

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$$-\frac{v_0^2}{2} = -g x_{\max}$$

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$$-\frac{v_0^2}{2} = -g x_{\max}$$

$$x_{\max} = \frac{v_0^2}{2g}.$$

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The acceleration is $a = -(mg + rv)/m$. The equation $v\dfrac{dv}{dx} = a$ becomes

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Solving, we get

$$x_{\max} = \frac{m}{r}\left[v_0 - \frac{mg}{r}\ln\left(1 + \frac{rv_0}{mg}\right)\right].$$

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Since $v > 0$, the acceleration is $a = -\frac{mg + kv^2}{m}$. The equation $v\frac{dv}{dx} = a$ becomes

$$\int_{v_0}^{0} \frac{v\,dv}{kv^2 + mg} = -\int_{0}^{x_{\max}} \frac{dx}{m}.$$

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$$x_{\max} = \frac{m}{2k} \ln\left(1 + \frac{kv_0^2}{mg}\right).$$