Math 211

Lecture #7

Mixing Problems

September 11, 2002

Mixing Problem #1

A tank originally holds 500 gallons of pure water. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 5 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.

Return

Model

- $S(t) =$ the amount of sugar in the tank in lbs.
- *Concentration* $=$ pounds per unit volume.
  - $c(t) = \frac{S(t)}{V} \frac{\text{lbs}}{\text{gal}}$.
- Modeling is easier in terms of the total amount, $S(t)$.
- Draw a picture.
- We must compute the rate of change of S in two ways.
  - The mathematical way: Rate of change $= dS/dt$.
  - The application way: This is where the real modeling takes place.

Return Problem

The Rate of Change of $S(t)$

- Balance Law:
  Rate of change = Rate in - Rate out
- Rate = volume rate × concentration
- For the problem
  - Rate in $= 5\frac{\text{gal}}{\text{min}} \times \frac{1}{2}\frac{\text{lb}}{\text{gal}} = 2.5\frac{\text{lb}}{\text{min}}$
  - Rate out $= 5\frac{\text{gal}}{\text{min}} \times \frac{S}{500}\frac{\text{lb}}{\text{gal}} = \frac{S}{100}\frac{\text{lb}}{\text{min}}$

Return

Solution

$$\frac{dS}{dt} = \text{Rate in - Rate out}$$
$$= 2.5 - \frac{S}{100}.$$

- The equation is linear.
- General solution: $S(t) = 250 + Ce^{-t/100}$.
- Particular solution: $S(t) = 250(1 - e^{-t/100})$.

Return Balance law

Other possible initial conditions

- There is initially 20 lbs of sugar in the tank.
- The concentration of sugar in the tank at $t = 0$ is 1 lb/gallon.

Solution Problem

Mixing Problem #2

A tank originally holds 500 gallons of sugar water with a concentration of $\frac{1}{10}$ lb/gal. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 10 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.

Solution

- Rate in $= 5\frac{\text{gal}}{\text{min}} \times \frac{1}{2}\frac{\text{lb}}{\text{gal}} = 2.5\frac{\text{lb}}{\text{min}}$
- Rate out $= 10\frac{\text{gal}}{\text{min}} \times \frac{S}{V}\frac{\text{lb}}{\text{gal}}$
  - $V(t) = 500 - 5t$.
  - Rate out $= \frac{10S}{500 - 5t}\frac{\text{lb}}{\text{min}}$

$$\begin{aligned}\frac{dS}{dt} &= \text{Rate in - Rate out}\\ &= 2.5 - \frac{2S}{100 - t},\end{aligned}$$

- General solution:

$$S(t) = 2.5(100 - t) + C(100 - t)^2.$$

- Particular solution:

$$S(t) = 2.5(100 - t) - \frac{(100 - t)^2}{50}.$$

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Conjectures, Theorems, and Proof

- A *conjecture* is a statement that we think is true.
- A *theorem* is a statement for which we have a logical *proof*.
  - A theorem contains:
    - *hypotheses* (the assumptions made)
    - and *conclusions*
  - The conclusions are guaranteed to be true if the hypotheses are true.
  - The implication goes only one way.

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Example of a "Theorem"

If it rains the sidewalks get wet.

- Hypothesis — *If it rains*
- Conclusion — *the sidewalks get wet*

Theorem

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Mathematics and Proof

- Theorems are proved by logical deduction.
- All of mathematics comes from a small number of very basic assumptions.
  - Called *axioms* or *postulates*.
- True of all parts of mathematics.
  - An algebraic derivation is an example of a proof.
- Definitions are not theorems.

Solving the Linear Equation

$$x' = a(t)x + f(t)$$

Four step process:

1. Rewrite as $x' - ax = f$.
2. Multiply by the integrating factor

   $$u(t) = e^{-\int a(t)\,dt}.$$

   Equation becomes $[ux]' = ux' - aux = uf$.
3. Integrate: $u(t)x(t) = \int u(t)f(t)\,dt + C$.
4. Solve for $x(t)$.

Return