Math 211

Lecture #9

Qualtitative Analysis

September 16, 2002

Uniqueness Theorem

Theorem: Suppose the function $f(t, y)$ and its partial derivative $\partial f/\partial y$ are continuous in the rectangle R in the ty-plane. Suppose that $(t_0, x_0) \in R$. Suppose that

$$x' = f(t, x) \quad \text{and} \quad y' = f(t, y),$$

and that

$$x(t_0) = y(t_0) = x_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$x(t) = y(t).$$

Theorem: Suppose $f(t,y)$, $\partial f/\partial y$ are continuous in the rectangle R. Let

$$M = \max_{(t,y)\in R} \left| \frac{\partial f}{\partial y}(t,y) \right|.$$

Suppose that (t_0, x_0) and (t_0, y_0) both lie in R, and

$$x' = f(t,x),\ x(t_0) = x_0 \ \&$$

$$y' = f(t,y),\ y(t_0) = y_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$|x(t) - y(t)| \le |x_0 - y_0| e^{M|t-t_0|}.$$

Continuity in Initial Conditions

- Inequality: $|x(t) - y(t)| \leq |x_0 - y_0| e^{M|t-t_0|}.$
- The good news:
  - By making sure that x_0 and y_0 are very close we can make the solutions $x(t)$ and $y(t)$ close for t in an interval containing t_0.
  - Solutions are *continuous in the initial conditions.*

Sensitivity with Respect to Initial Conditions

- Inequality: $|x(t) - y(t)| \le |x_0 - y_0| e^{M|t-t_0|}$.
- The bad news:
  - As $|t - t_0|$ increases the RHS grows exponentially.
  - Over long intervals in t the solutions can get very far apart. Solutions are *sensitive to initial conditions.*

Qualitative Analysis of Autonomous Equations

- Ways to discover the properties of solutions without solving the equation.
  - What happens to solutions $y(t)$ as $t \to \infty$.
- Properties of autonomous equations, $y' = f(y)$.
  - The direction field does not depend on t.
  - Solution curves can be translated left and right to get other solution curves. I.e., if $y(t)$ a solution so is $y_1(t) = y(t + c)$ for any constant c.

Equilibrium Points & Solutions

Autonomous equation: $y' = f(y)$.

- Equilibrium point: $f(y_0) = 0$.
- Equilibrium solution: $y(t) = y_0$.
- Example: $y' = \sin y$
  - $\sin y = 0 \quad \Leftrightarrow \quad y = k\pi, \quad k = 0, \pm 1, \ldots$
  - $y' = \sin y$ has infinitely many equilibrium solutions:

$$y_k(t) = k\pi \quad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Between the Equilibrium Points

Example: $y' = \sin y$.

- $0 < y < \pi$
  - $y'(t) = \sin y(t) > 0 \Rightarrow y(t)$ is increasing
  - By uniqueness, $0 < y(t) < \pi$ for all t.
  - $\Rightarrow y(t) \nearrow \pi$ as $t \to \infty$ and $y(t) \searrow 0$ as $t \to -\infty$
- $-\pi < y < 0$
  - $y'(t) = \sin y(t) < 0 \Rightarrow y(t)$ is decreasing
  - By uniqueness, $0 > y(t) > -\pi$ for all t.
  - $\Rightarrow y(t) \searrow -\pi$ as $t \to \infty$ and $y(t) \nearrow 0$ as $t \to -\infty$

Stable & Unstable EPs

An equilibrium point y_0 is

- *asymptotically stable* if all solutions starting near y_0 converge to y_0 as $t \longrightarrow \infty$.
- *unstable* if there are solutions starting arbitrarily close to y_0 which move away from y_0 as t increases.
- There are 4 possibilities:

A Phase Line for $y' = f(y)$

- A *phase line* is a y-axis, showing
  - the equilibrium points and
  - the direction of the flow between the equilibrium points.
- Examples:
  - The y-axis in the plot of $y \to f(y)$.
  - The y-axis in the ty-plane where solutions are plotted.

Example – Terminal Velocity

- Assume the magnitude of the resistance is proportional to the square of the velocity:

$$v' = -g - k|v|v/m$$

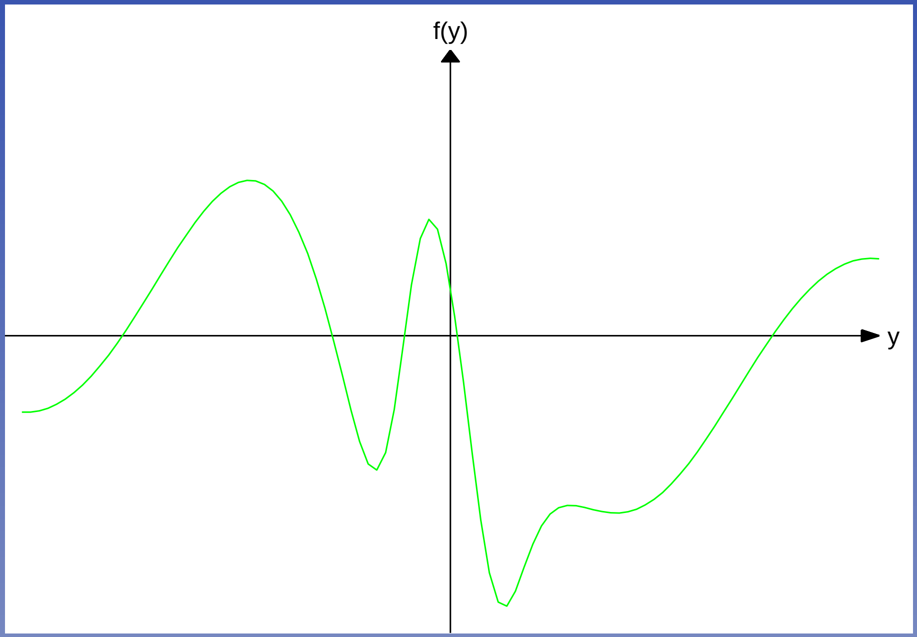
- One equilibrium point at

$$v_{\text{term}} = -\sqrt{\frac{mg}{k}}.$$

- v_{term} is asymptotically stable.

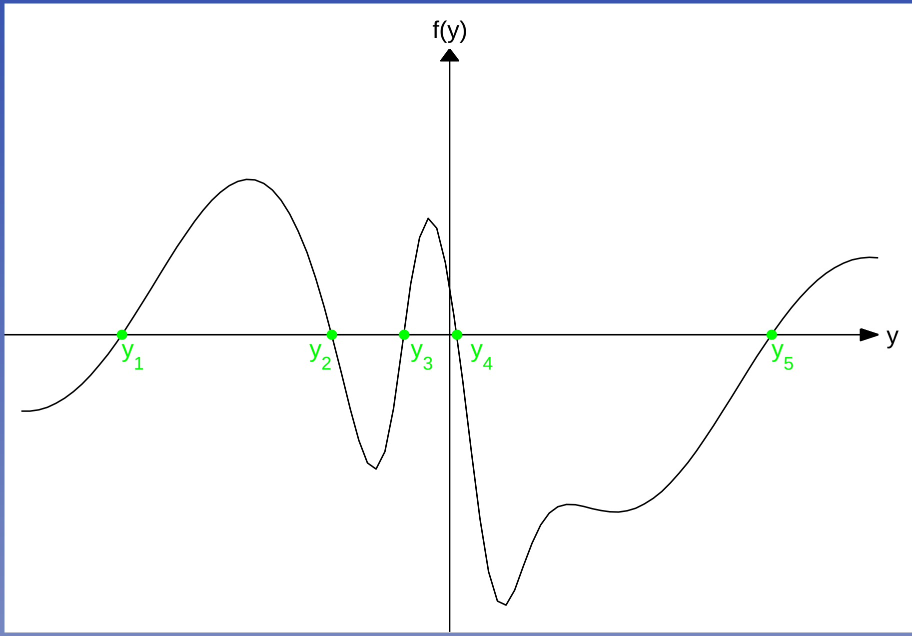
Qualitative Analysis of $y' = f(y)$.

1. Graph $y \to f(y)$.



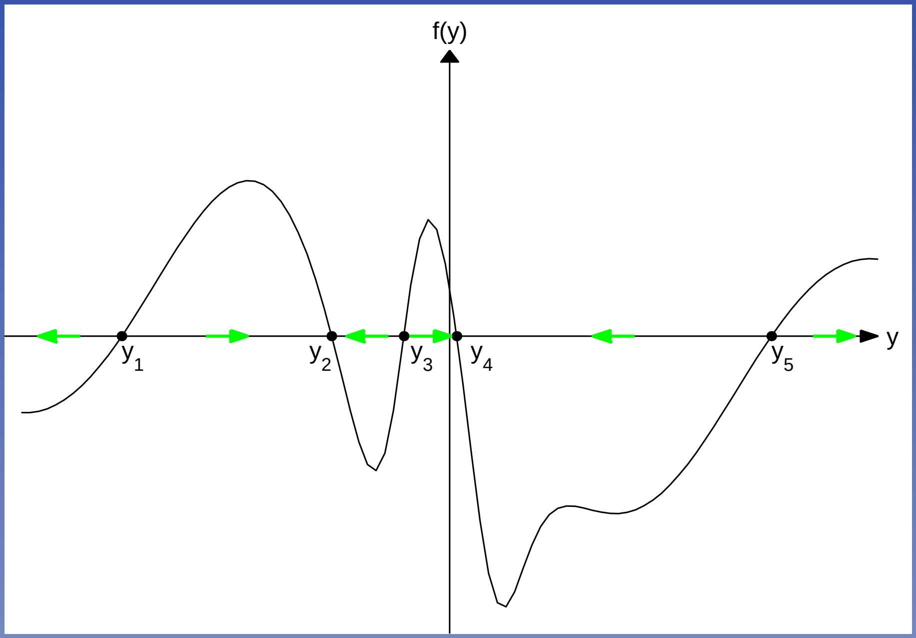
Qualitative Analysis of $y' = f(y)$.

2. Find the equilibrium points where $f(y) = 0$.



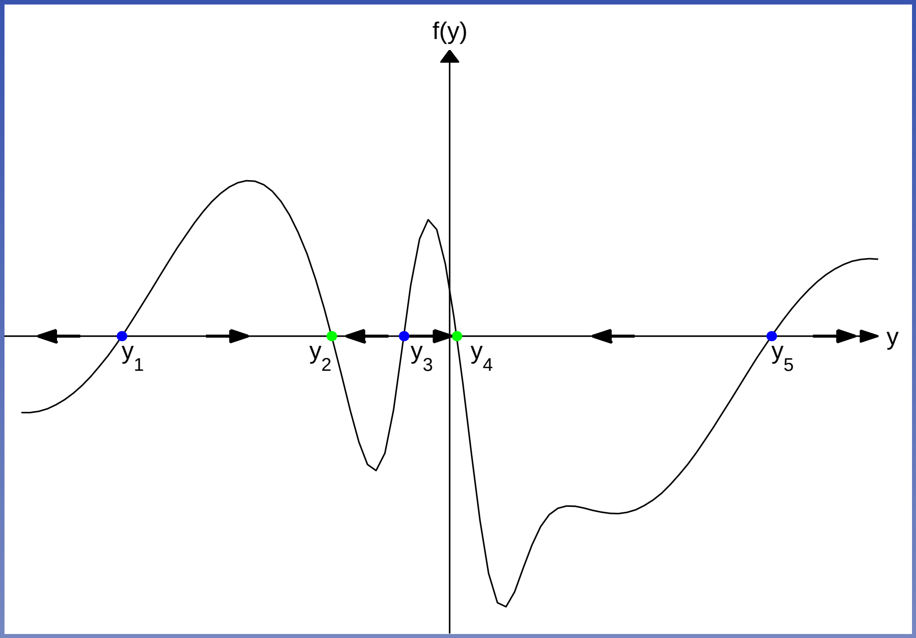
Qualitative Analysis of $y' = f(y)$.

3. Determine the behavior between eq. pts.



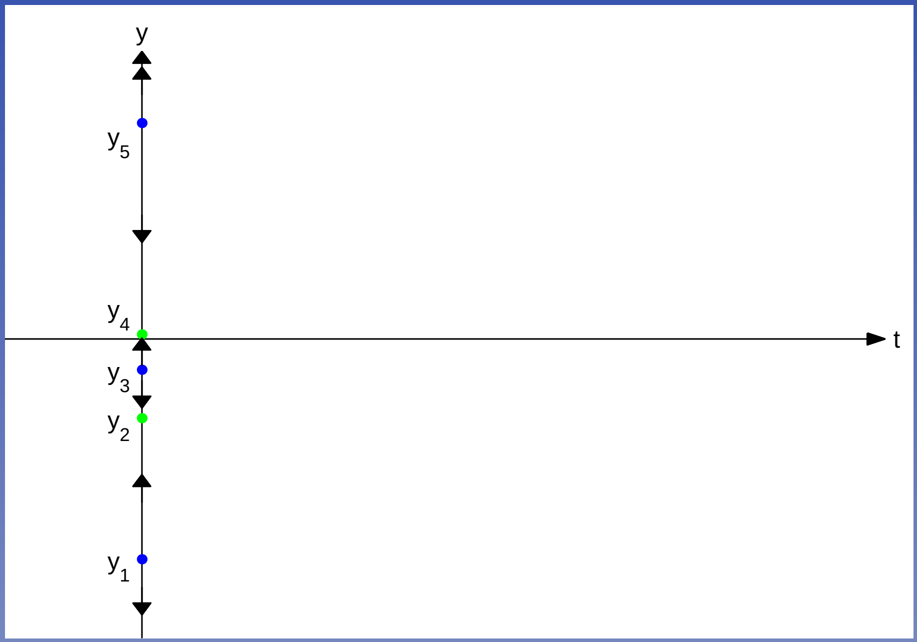
Qualitative Analysis of $y' = f(y)$.

4. Analyze the equilibrium points.



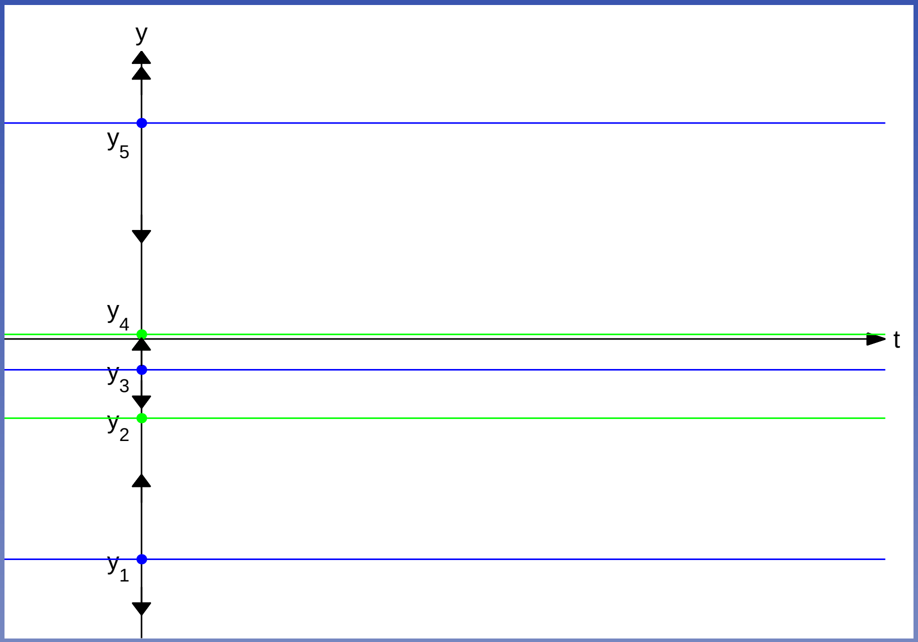
Qualitative Analysis of $y' = f(y)$.

5. Transfer the phase line to ty-space.



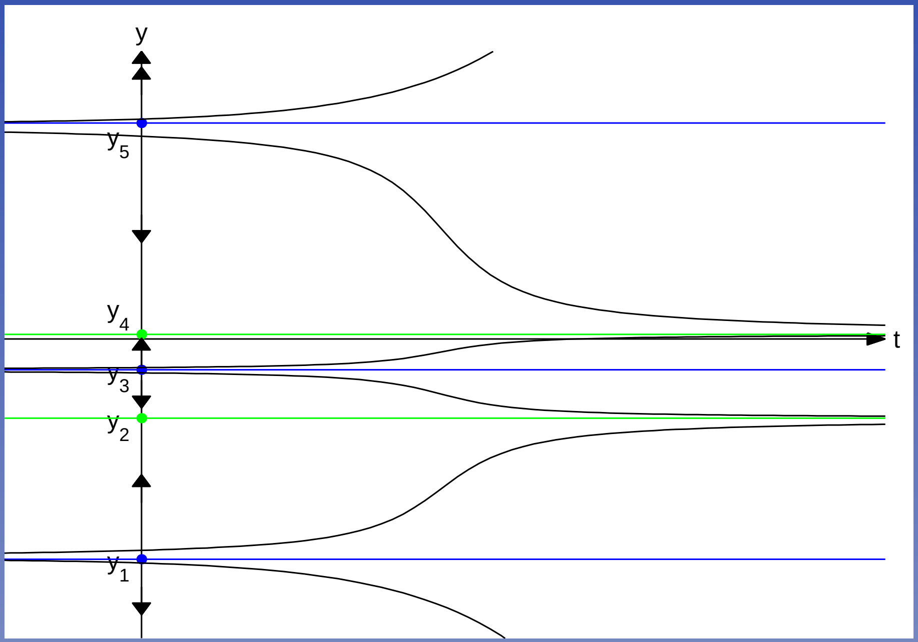
Qualitative Analysis of $y' = f(y)$.

6. Plot the equilibrium solutions.



Qualitative Analysis of $y' = f(y)$.

7. Plot other solutions approximately.



Seven Steps

1. Graph $y \to f(y)$.
2. Find the equilibrium points where $f(y) = 0$.
3. Determine the behavior between eq. pts.
4. Analyze the equilibrium points.
5. Transfer the phase line to ty-space.
6. Plot the equilibrium solutions.
7. Plot other solutions approximately.