Math 211

Lecture #10

Population Models

September 18, 2002

Modeling Population

- Assume population changes due to births and deaths only.
- Births are roughly proportional to population, $B = bP$
  - b is the *birth rate*. It is the average number of births per individual in one unit of time.
- Deaths are roughly proportional to population, $D = dP$.
  - d is the *death rate*. It is the probability that any one individual will die in one unit of time.

Modeling Population (cont.)

- Rate of change = births − deaths

$$\frac{dP}{dt} = B - D = bP - dP = rP$$

  - $r = b - d$ is the *reproductive rate*.
  - In general, b and d, and therefore r, are not constants.
    - They can depend on P, and perhaps also on t.

The Malthusian Model

- If there exist sufficient resources in term of nutrients and space, b and d will be almost constant. Then the reproductive rate $r = b - d$ is almost a constant.
- If r is constant we have the *Malthusian model*.

$$\frac{dP}{dt} = rP \quad \text{with} \quad P(0) = P_0$$

- Solution: $P(t) = P_0 e^{rt}$
  - If $r = b - d > 0$, $P(t)$ grows exponentially.
  - If $r = b - d < 0$, $P(t)$ decays exponentially.

The Malthusian Model (cont.)

Under what circumstances could the Malthusian model be a good model?

- Requires unlimited resources.
  - OK in laboratory experiments with small populations.
- Populations always outgrow the Malthusian model. This was the point that was made by Malthus.

The Logistic Model

- As the population increases individuals compete for resources — for food and for space.
- The birth rate b is the average number of births per individual in one unit of time.
- As $P \nearrow$, $b \searrow$ because of competition.
  - Competition results from encounters.
  - The number of encounters by one individual is roughly proportional to P.
  - $\Rightarrow$ decrease in the birth rate is $\sim P$
  - Assume that $b = b_0 - b_1 P$

The Logistic Model (cont.)

- Increase in the death rate d is $\sim P$
  - Assume that $d = d_0 + d_1 P$
- The reproductive rate is

$$r = b - d = (b_0 - b_1 P) - (d_0 + d_1 P) = r_0 - r_1 P$$

- The result is the *logistic model*

$$\begin{aligned} \frac{dP}{dt} &= rP = (r_0 - r_1 P)P \\ &= r_0 \left(1 - \frac{P}{K}\right) P \quad (K = r_0/r_1) \end{aligned}$$

Analysis of the Logistic Model

$$\frac{dP}{dt} = r_0 \left(1 - \frac{P}{K}\right) P$$

- Equation is autonomous.
- Equilibrium points are 0 & K.
- 0 is unstable, K is asymptotically stable.
- Any positive solution $P(t) \to K$ as $t \to \infty$.
  - K is the *carrying capacity.*
  - r_0 is the *reproductive rate at small populations.*

Solution of The Logistic Model

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P \quad \text{with} \quad P(0) = P_0$$

- Solution:

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

Estimating Parameters

- Malthusian model $P' = rP$

$$P(t) = P_0 e^{rt}$$

  - Two parameters P_0 and r.
  - Two measurements or observations needed to find the values of P_0 and r.
  - It is better to use all of the data and use least squares (linear regression).

Estimating Parameters

- Logistic model $P' = r(1 - P/K)P$

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

  - Three parameters, P_0, r, and K.
  - Three measurements or observations needed to find the values of P_0, r, and K.
  - It is better to use all of the data and use least squares. (Nonlinear regression)

Modelling

- Two ways to write the rate of change of something, e.g., of a population P
  - The mathematical way is the derivative, $\frac{dP}{dt}$.
  - The other way involves scientific analysis.,

$$r\left(1-\frac{P}{K}\right)P.$$

- Setting the two equal gives a differential equation model, in this case the logistic model

$$\frac{dP}{dt}=r\left(1-\frac{P}{K}\right)P$$

Efficacy of the Logistic Model

- Does a very good job of modeling the growth of populations under controlled circumstances.
  - In laboratory experiments.
  - In other circumstances when the situation does not change.
- For human populations the model always breaks down.
  - Other factors become important, such as immigration , the advance of technology, and changing habits of life.