

Math 211

Lecture #11
Financial Models

September 20, 2002

Compound Interest

Put some money into an account that returns a percentage each year, compounded continuously. How will it grow?

- $P(t)$ is the principal balance in the account, measured in \$1000.
- "Some money" is $P(0) = P_0$.
- "Returns a percentage" is $r\%$ /year.
- "Some time later" is measured in years.
- "Compounded continuously" $\Rightarrow P' = rP$.

Return

Compound Interest (cont.)

- Solution

$$P(t) = P_0 e^{rt}$$

- The principal grows exponentially.
- If $r = 8\%$, then after 20 years

$$\begin{aligned} P(20) &= P_0 e^{0.08 \times 20} \\ &= 4.953 P_0 \end{aligned}$$

- After 40 years $P(40) = 24.5325 P_0$.

Return

Returns on Investments

What rates of return can we expect?

- Checking accounts — 0 – 3%.
- Money market accounts — 1/4 – 3%.
- Certificates of deposit (3 years) 3 – 4 %.
- Industrial bonds — 5.3% (average from 1926 – 2001).
- Stocks — 10.7% (average from 1926 – 2001).

Return

Example

Retirement Account

- Set up a retirement account by investing an initial amount. In addition, deposit a fixed amount each year until you retire. Assume it returns a percentage each year, compounded continuously. How much is there some time later?
 - ♦ “A fixed amount each year” is D , measured in \$1,000 each year. We assume this is invested continuously.

Return

Retirement Account (cont.)

- The model is

$$P' = rP + D.$$

- Solution

$$P(t) = P_0 e^{rt} + \frac{D}{r} [e^{rt} - 1].$$

Return

Definitions

Example of a Retirement Account

- Suppose you start with an investment of \$1,000 at the age of 25, and invest \$100 each month until you retire at 65. The account returns 8% per year. How much is in the retirement account when you retire?
 - ♦ $P_0 = 1000$, $D = 100 \times 12 = 1200$, $r = 8\% = 0.08$.
- At 65 the principal is \$377,521.
- Is this enough to retire on?

Return

Model

Retirement Planning

- If you need a certain income after you retire, how much must you have in your retirement account when you retire?
 - ♦ "Certain income" is I (in \$1000/year) withdrawn from the account.
 - ♦ "How much" is the amount P_0 in the account at retirement.
 - ♦ The account still grows due to its return at $r\%$ /year.

Return

Example

Retirement Planning (cont.)

- The model is

$$P' = rP - I, \quad P(0) = P_0.$$

- Solution $P(t) = P_0 e^{rt} - \frac{I}{r}[e^{rt} - 1]$.
- We are given I , r , & $P(t_d)$.
- We need to compute P_0 .

Return

Definitions

Retirement Planning – Example 1

- If you will need an income of \$75,000 for 30 years after retirement and your account returns 6%, your account balance at retirement should be

\$1,043,000.

- How are you going to save over a million dollars?

Return

Retirement Planning (cont.)

- Instead of investing a fixed amount each month, it would be more realistic to invest a percentage of your salary. What should this percentage be in order to accumulate an adequate investment balance? Include the effect of inflation.
- Your starting salary is S_0 .
- Assume it will increase at $s\%$ per year.
 - ♦ Then $S' = sS$, or $S(t) = S_0e^{st}$.

Return

Retirement Planning (cont.)

- The model for the growth of the retirement account is

$$P' = rP + \lambda S_0 e^{st} \quad \text{with} \quad P(0) = P_0.$$

- Solution

$$P(t) = P_0 e^{rt} + \frac{\lambda S_0}{r - s} [e^{rt} - e^{st}].$$

Return

Definition

Retirement Planning – Example 2

- Assume
 - ♦ $P_0 = \$1,000$ and $r = 8\%$
 - ♦ $S_0 = \$35,000$ and $s = 4\%$
 - ▶ Notice that $S(40) = \$173,356$.
 - ♦ Need a retirement income of $\$150,000$.
 - ▶ Aim for a balance at retirement of $\$2,000,000$.
- Requires $\lambda = 11.53\%$.

Return

Model

Other Strategies

- Delayed gratification. Deposit a percentage of your salary that starts at $\lambda\%$, and decays linearly to 0 over 40 years.

$$P' = rP + \lambda(1 - t/40)S_0e^{st}$$

- Immediate gratification. Deposit a percentage of your salary that starts at 0 and grows linearly over 40 years to $\lambda\%$.

$$P' = rP + \frac{\lambda t}{40}S_0e^{st}$$

Model