Math 211

Lecture #11

Financial Models

September 20, 2002

Compound Interest

Put some money into an account that returns a percentage each year, compounded continuously. How will it grow?

- $P(t)$ is the principal balance in the account, measured in \$1000.
- "Some money" is $P(0) = P_0$.
- "Returns a percentage" is $r\%/\text{year}$.
- "Some time later" is measured in years.
- "Compounded continuously" $\Rightarrow$ $P' = rP$.

Compound Interest (cont.)

- Solution

$$P(t) = P_0 e^{rt}$$

- The principal grows exponentially.
- If $r = 8\%$, then after 20 years

$$P(20) = P_0 e^{0.08 \times 20}$$
$$= 4.953\, P_0$$

- After 40 years $P(40) = 24.5325\, P_0$.

Return

Returns on Investments

What rates of return can we expect?

- Checking accounts — 0 – 3%.
- Money market accounts — 1/4 – 3%.
- Certificates of deposit (3 years) 3 – 4 %.
- Industrial bonds — 5.3% (average from 1926 – 2001).
- Stocks — 10.7% (average from 1926 – 2001).

Retirement Account

- Set up a retirement account by investing an initial amount. In addition, deposit a fixed amount each year until you retire. Assume it returns a percentage each year, compounded continuously. How much is there some time later?
  - "A fixed amount each year" is D, measured in \$1,000 each year. We assume this is invested continuously.

Retirement Account (cont.)

- The model is

$$P' = rP + D.$$

- Solution

$$P(t) = P_0 e^{rt} + \frac{D}{r}[e^{rt} - 1].$$

Example of a Retirement Acount

- Suppose you start with an investment of \$1,000 at the age of 25, and invest \$100 each month until you retire at 65. The account returns 8% per year. How much is in the retirement account when you retire?
  - $P_0 = 1000$, $D = 100 \times 12 = 1200$, $r = 8\% = 0.08$.
- At 65 the principal is \$377,521.
- Is this enough to retire on?

Retirement Planning

- If you need a certain income after you retire, how much must you have in your retirement account when you retire?
  - "Certain income" is I (in \$1000/year) withdrawn from the account.
  - "How much" is the amount P_0 in the account at retirement.
  - The account still grows due to its return at r%/year.

Retirement Planning (cont.)

- The model is

$$P' = rP - I, \quad P(0) = P_0.$$

- Solution $P(t) = P_0 e^{rt} - \frac{I}{r}[e^{rt} - 1]$.
- We are given I, r, & $P(t_d)$.
- We need to compute P_0.

Retirement Planning – Example 1

- If you will need an income of \$75,000 for 30 years after retirement and your account returns 6%, your account balance at retirement should be

  \$1,043,000.

- How are you going to save over a million dollars?

Retirement Planning (cont.)

- Instead of investing a fixed amount each month, it would be more realistic to invest a percentage of your salary. What should this percentage be in order to accumulate an adequate investment balance? Include the effect of inflation.
- You starting salary is S_0.
- Assume it will increase at s% per year.
  - Then $S' = sS$, or $S(t) = S_0 e^{st}$.

Return

Retirement Planning (cont.)

- The model for the growth of the retirement account is

$$P' = rP + \lambda S_0 e^{st} \quad \text{with} \quad P(0) = P_0.$$

- Solution

$$P(t) = P_0 e^{rt} + \frac{\lambda S_0}{r - s}\left[e^{rt} - e^{st}\right].$$

Retirement Planning – Example 2

- Assume
  - $P_0 = \$1{,}000$ and $r = 8\%$
  - $S_0 = \$35{,}000$ and $s = 4\%$
    - Notice that $S(40) = \$173{,}356$.
  - Need a retirement income of \$150,000.
    - Aim for a balance at retirement of \$2,000,000.
- Requires $\lambda = 11.53\%$.

Other Strategies

- Delayed gratification. Deposit a percentage of your salary that starts at $\lambda\%$, and decays linearly to 0 over 40 years.

$$P' = rP + \lambda(1 - t/40)S_0 e^{st}$$

- Immediate gratification. Deposit a percentage of your salary that starts at 0 and grows linearly over 40 years to $\lambda\%$.

$$P' = rP + \frac{\lambda t}{40} S_0 e^{st}$$