

Math 211

Lecture #12

Numerical Methods — Euler's Method

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Numerical Methods

- A numerical “solution” is not a solution.
- It is a discrete approximation to a solution.
- We make an error on purpose to enable us to compute an approximation.
- Extremely important to understand the size of the error.

Numerical Approximation

To numerically “solve” $y' = f(t, y)$ with $y(a) = y_0$ on the interval $[a, b]$, we find

- a discrete set of points

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b$$

- and values $y_0, y_1, y_2, \dots, y_{N-1}, y_N$ with y_j approximately equal to $y(t_j)$.
- Making an error $E_j = y(t_j) - y_j$.

Types of Solvers

- We will discuss and use four solvers
 - ◆ Euler's method,
 - ◆ second order Runge-Kutta,
 - ◆ fourth order Runge-Kutta,
 - ◆ and ode45.
- Everything works for first order systems almost without change.

Euler's Method

- Problem: Solve (*approximately*)

$$y' = f(t, y) \quad \text{with} \quad y(a) = y_0$$

on the interval $[a, b]$.

- Discrete set of values of t .
 - ◆ $t_0 = a$, *fixed step size* $h = (b - a)/N$.
 - ◆ $t_1 = t_0 + h$, $t_2 = t_1 + h = t_0 + 2h$, etc,
 - ◆ $t_N = a + Nh = b$

Euler's Method – First Step

- At each step approximate the solution curve by the tangent line.
- First step:
 - ◆ $y(t_0 + h) \approx y(t_0) + y'(t_0)h. \quad t_1 = t_0 + h$
 - ◆ $y(t_1) \approx y_0 + f(t_0, y_0)h.$
 - ◆ Set $y_1 = y_0 + f(t_0, y_0)h,$ so $y(t_1) \approx y_1.$

Euler's Method – Second Step

- At each step use the tangent line.
- Second step – start at (t_1, y_1) .
 - ◆ New solution \tilde{y} with initial value $\tilde{y}(t_1) = y_1$.
 - ◆ $\tilde{y}(t) \approx \tilde{y}(t_1) + \tilde{y}'(t_1)(t - t_1), \quad t_2 = t_1 + h$
 - ◆ $\tilde{y}(t_2) \approx y_1 + f(t_1, y_1)h.$
 - ◆ Set $y_2 = y_1 + f(t_1, y_1)h,$ so $y(t_2) \approx \tilde{y}(t_2) \approx y_2.$

Euler's Method – Algorithm

Input t_0 and y_0 .

for $k = 1$ to N set

$$y_k = y_{k-1} + f(t_{k-1}, y_{k-1})h$$

$$t_k = t_{k-1} + h$$

Thus,

$$y_1 = y_0 + f(t_0, y_0)h \quad \text{and} \quad t_1 = t_0 + h$$

$$y_2 = y_1 + f(t_1, y_1)h \quad \text{and} \quad t_2 = t_1 + h$$

$$y_3 = y_2 + f(t_2, y_2)h \quad \text{and} \quad t_3 = t_2 + h$$

etc.

MATLAB routine eulerdemo.m

- Demonstrates truncation error.
- Demonstrates how truncation error can propagate exponentially.
- Demonstrates how the total error is the sum of propagated truncation errors.

Error Analysis – First Step

- Euler's approximation

$$y_1 = y_0 + f(t_0, y_0)h; \quad t_1 = t_0 + h$$

- Taylor's theorem

$$y(t_1) = y(t_0 + h) = y(t_0) + y'(t_0)h + R(h)$$

$$|R(h)| \leq Ch^2$$

- $y(t_1) - y_1 = R(h)$
- The **truncation error** at each step is the same as the Taylor remainder, and $|R(h)| \leq Ch^2$.

Error Analysis

- There are $N = (b - a)/h$ steps.
- **Truncation error** can grow exponentially.
- Computation shows that

$$\text{Maximum error} \leq C \left(e^{L(b-a)} - 1 \right) h,$$

where C & L are constants that depend on f .

- Good news: the error decreases as h decreases.
- Bad news: the error can get exponentially large as the length of the interval [i.e., $b-a$] increases.

MATLAB routine eul.m

Syntax: `[t,y] = eul(derfile,[t0,tf],y0,h);`

- `derfile` - derivative m-file defining the equation.
- `t0` - initial time; `tf` - final time.
- `y0` - initial value.
- `h` - step size.

Derivative m-file

The **derivative m-file** describes the differential equation.

- Example: $y' = y^2 - t$
- Derivative m-file:

```
function ypr = george(t,y)

ypr = y^2 - t;
```

- Save as `george.m`.

Use of eul.m

- Solve $y' = y^2 - t$.
- Use the derivative m-file `george.m`.
- Use $t_0 = 0$, $t_f = 10$, $y_0 = 0.5$, and several step sizes.
- Syntax: `[t,y] = eul('george',[0,10],0.5,h);`

Experimental Error Analysis

- IVP $y' = \cos(t)/(2y - 2)$ with $y(0) = 3$
- Exact solution: $y(t) = 1 + \sqrt{4 + \sin t}$.
- Solve using Euler's method and compare with the exact solution.
- Do this for several step sizes.

Derivative m-file `ben.m`

```
function yprime = ben(t,y)
yprime = cos(t)/(2*y-2);
```

M-file batch.m

```
[teuler,yeuler]=eul('ben',[0,3],3,h);  
t=0:0.05:3;  
y=1+sqrt(4+sin(t));  
plot(t,y,teuler,yeuler,'o')  
legend('Exact','Euler')  
shg  
z=1+sqrt(4+sin(teuler));  
maxerror=max(abs(z-yeuler))
```